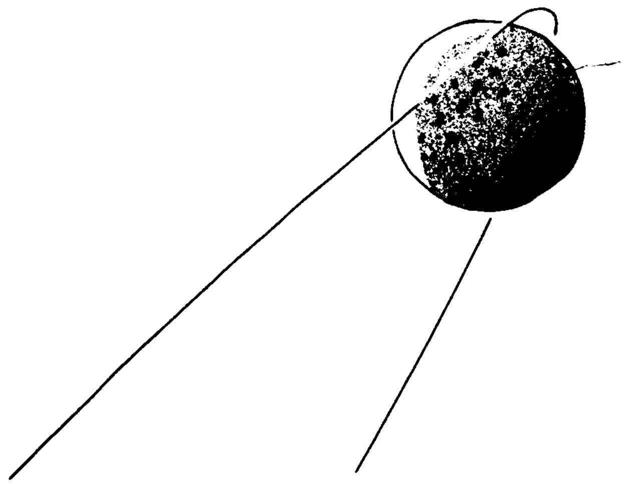

OPTIMIZATION OF NONLINEAR SYSTEMS WITH INEQUALITY CONSTRAINTS

BY

GARY J. LASTMAN

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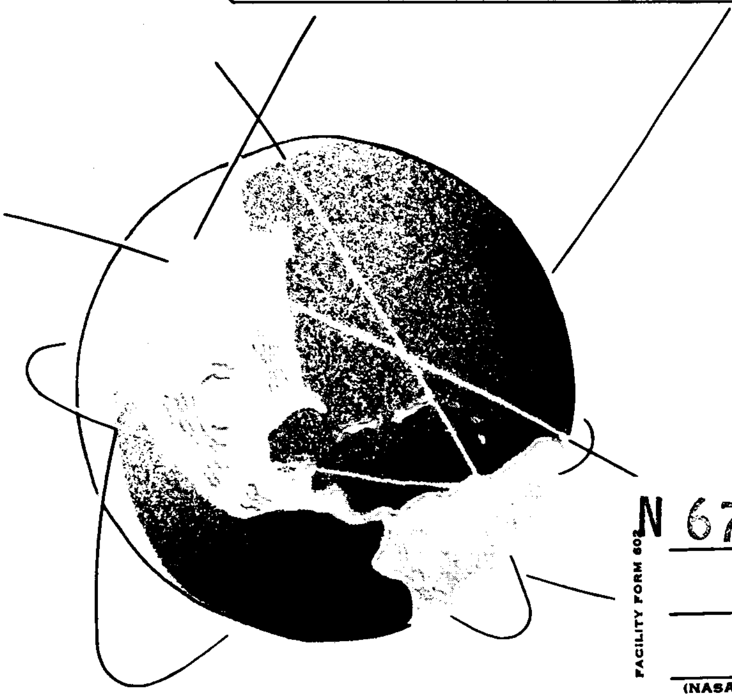

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OPTIMIZATION OF NONLINEAR SYSTEMS WITH INEQUALITY CONSTRAINTS

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PREFACE

There are many physical systems whose action can be controlled and whose performance is to be optimized in some sense. The study of the means of attaining the desired optimum behavior of a system constitutes a basic problem in optimization theory. Often the action of a system may be constrained by certain physical limitations. In this investigation the problem of optimizing the performance of a constrained system is examined. Only systems described by a set of first order ordinary differential equations, and physical limitations which can be represented by algebraic inequalities are discussed.

Acknowledgement is given to Dr. B. D. Tapley who supervised the research in this dissertation; to Dr. W. T. Fowler for many long and interesting discussions; and to Dr. P. L. Odell and Dr. E. A. Ripperger who served on the doctoral committee.

G. J. L.

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CHAPTER 1

INTRODUCTION

Many dynamical systems of engineering interest are described by a set of first order ordinary differential equations whose solutions must satisfy specified initial and terminal conditions. An example of such a dynamical system is the trajectory of a missile which is required to intercept a moving target. The process of optimizing the motion of a dynamical system consists of the selection of certain input variables which appear in the derivative functions, in such a way that the performance of the given system is optimal in some sense. The input variables are known as control variables. The optimization problem can be restated in terms of the minimization of some quantity (usually referred to as the performance index) subject to (1) the satisfaction of a set of differential equations which describe the dynamical system, (2) specified conditions which the system must satisfy at the initial and terminal times, and (3) any additional conditions which may be imposed. The third point is important because physical limitations on the motion of the dynamical system may be included as additional conditions for the problem. An example is the limited thrust magnitude on a variable-thrust rocket. In this work, only one type of formulation of a physical limitation is considered—the inequality constraint. There are many different physical limitations on the motion of a dynamical system that can be represented by inequality constraints [6,9,15,35,37,38]. Included in these is the bounded thrust magnitude of a variable-thrust rocket.

Optimization problems with inequality constraints have been the subject of extensive theoretical studies involving the analysis of general problems by the techniques of the Calculus of Variations and Optimal Control Theory, [2,3,21,22,23,24,43,46,48]. The earliest work on this topic, Valentine [48], treated the Problem of Lagrange with inequality constraints from the viewpoint of the Calculus of Variations. In the book by Pontryagin et al. [43], the Maximum Principle of Optimal Control Theory was used. Berkovitz [2,3] reduced a general problem formulated in Optimal Control Theory to the Problem of Bolza in the Calculus of Variations in order to use the known results of the Bolza Problem. In each of these theoretical studies, necessary conditions which the solution to the optimization problem must satisfy are obtained.

A knowledge of these necessary conditions is usually insufficient to produce the solution to the optimization problem. However, the known conditions may be used to reduce the optimization problem to a two-point boundary value problem which then can be resolved by an iterative numerical procedure. The engineer is concerned not only with the properties of an optimal solution but also with the problem of obtaining such a solution. Hence, the purpose of this investigation can be stated as follows: (1) to study the effect of inequality constraints on the optimization of dynamical systems described by first order ordinary differential equations, which are usually nonlinear, and (2) to provide a computational algorithm that can be used to obtain a solution to the resulting two-point boundary value problem.

In Chapter 2 an optimization problem with inequality constraints is formulated. Then the properties of two general forms of inequality constraints are discussed. This discussion leads to a problem reformulation, which is treated in Chapter 3. Computational algorithms based on a new perturbation method for inequality-constrained nonlinear problems are discussed in Chapter 4. The numerical solution of a constrained nonlinear problem is presented in Chapter 5. This example problem is a mathematical model of an Earth-Mars transfer trajectory in three dimensions, with inequality constraints as added side conditions.

The notation to be used in the following chapters is given below.

(1) $\frac{\partial A}{\partial B} = A_B$.

(2) A superscript T denotes transpose; a superscript -1 denotes inverse.

(3) If X is an n-dimensional vector, then the norm of X is

$$||X|| = \max_{1 \leq j \leq n} |X_j| .$$

(4) If Q is a scalar then

$$\frac{\partial Q}{\partial X} = Q_X = [\frac{\partial Q}{\partial X_1}, \frac{\partial Q}{\partial X_2}, \ldots, \frac{\partial Q}{\partial X_n}]$$

where X is an n-dimensional column vector.

(5) If Y is an m-dimensional column vector and X is an n-dimensional column vector, then

$$\frac{\partial Y}{\partial X} = Y_X = \begin{bmatrix} \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial Y_m}{\partial X_1} & \cdots & \frac{\partial Y_m}{\partial X_n} \end{bmatrix}.$$

(6) If Q is a scalar and X and Y are defined as in (5) then

$$\frac{\partial^2 Q}{\partial X \partial Y} = Q_{XY} = \begin{bmatrix} \frac{\partial^2 Q}{\partial X_1 \partial Y_1} & \frac{\partial^2 Q}{\partial X_1 \partial Y_2} & \cdots & \frac{\partial^2 Q}{\partial X_1 \partial Y_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 Q}{\partial X_n \partial Y_1} & \frac{\partial^2 Q}{\partial X_n \partial Y_2} & \cdots & \frac{\partial^2 Q}{\partial X_n \partial Y_m} \end{bmatrix}.$$

From this definition it follows that

$$Q_{XY} = (Q_{YX})^T$$

provided Q has continuous second partial derivatives at the point at which Q_{XY} and Q_{YX} are evaluated.

(7) Let t be a continuous scalar variable and let $W = W(t)$. Then,

$$W(t_1^+) = \lim_{\substack{t \to t_1 \\ t > t_1}} W(t)$$

$$W(t_1^-) = \lim_{\substack{t \to t_1 \\ t < t_1}} W(t)$$

If W is either a vector or a matrix the above expression applies to each component of the vector and to each element of the matrix.

(8) $\dot{X}$ denotes $\frac{dX}{dt}$.

(9) $(\;)_t$ denotes $(\;)$ evaluated at t.

(10) The variation of X, δX, is denoted by x. A total change in X is denoted by ΔX. Generally, $\Delta X = x + \dot{X}\Delta t$.

CHAPTER 2

ON INEQUALITY CONSTRAINTS

Consider the following optimization problem involving inequality constraints: determine the control program $U(t)$ on the interval $t_0 \le t \le t_N$ so as to minimize the functional

$$\Gamma = G(X(t_N),t_N) + \int_{t_0}^{t_N} Q(X,U,t)dt \tag{2.1}$$

while satisfying the conditions

$$\dot{X} = F(X,U,t) \tag{2.2}$$

$$C_i(X,U,t) \le 0 \quad (i = 1,2,\ldots,\alpha) \tag{2.3}$$

$$S_j(X,t) \le 0 \quad (j = 1,2,\ldots,\beta) \tag{2.4}$$

on the interval $t_0 \le t \le t_N$, and

$$L(X(t_N),t_N) = 0 \; ; \tag{2.5}$$

where t_0 is known and $X(t_0)$ is specified, i.e.,

$$X(t_0) = X^{\circ} \tag{2.6}$$

The following definitions are used in Relationships (2.1) through (2.6).

$$X = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \cdot \\ \cdot \\ \cdot \\ X_n(t) \end{bmatrix}$$, an n-dimensional vector of state variables;

$$U = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \cdot \\ \cdot \\ \cdot \\ U_m(t) \end{bmatrix}$$, an m-dimensional vector of control variables;

$$F = \begin{bmatrix} F_1(X,U,t) \\ F_2(X,U,t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(X,U,t) \end{bmatrix}$$, a specified n-dimensional vector function of X,U and t;

$$L = \begin{bmatrix} L_1(X(t_N),t_N) \\ L_2(X(t_N),t_N) \\ \cdot \\ \cdot \\ \cdot \\ L_\ell(X(t_N),t_N) \end{bmatrix}$$, a specified ℓ-dimensional vector of terminal constraint functions;

$G(X(t_N),t_N)$, a specified scalar function of the terminal values of X and t;

$Q(X,U,t)$, a specified scalar function of X, U, and t;

$$C = \begin{bmatrix} C_1(X,U,t) \\ C_2(X,U,t) \\ \cdot \\ \cdot \\ \cdot \\ C_\alpha(X,U,t) \end{bmatrix}$$, a specified α-dimensional vector of inequality constraint functions where each function explicitly contains the control U;

$$S = \begin{bmatrix} S_1(X,t) \\ S_2(X,t) \\ \cdot \\ \cdot \\ \cdot \\ S_\beta(X,t) \end{bmatrix}$$, a specified β-dimensional vector of inequality constraint functions which do not contain the control U;

and t, the scalar independent variable, hereafter referred to as "time".

In many problems inequality constraints of the form

$$g(t_N) + \int_{t_0}^{t_N} v(X,U,t)dt \leq 0 \tag{2.7}$$

or

$$h(t_N) + \int_{t_0}^{t_N} w(X,t)dt \leq 0 \tag{2.8}$$

are given. These integral or isoperimetric constraints can be reduced to terminal inequality constraints by introducing two new state variables, X_{n+1} and X_{n+2}. The new state variables satisfy the following differential equations:

$$\dot{X}_{n+1} = v(X,U,t) \;, \quad X_{n+1}(t_0) = 0$$

$$\dot{X}_{n+2} = w(X,t) \;, \quad X_{n+2}(t_0) = 0 \;.$$

Then the Inequalities (2.7) and (2.8) are equivalent to the inequalities

$$L_{\ell+1} = g(t_N) + X_{n+1}(t_N) \leq 0 \tag{2.7'}$$

and

$$L_{\ell+2} = h(t_N) + X_{n+2}(t_N) \leq 0 \qquad (2.8')$$

respectively. If the Inequalities (2.7') and (2.8') are expressed as equalities by using the method of Valentine [48], (see Chapter 3), then $L_{\ell+1}$ and $L_{\ell+2}$ can be treated as terminal constraint functions. Therefore, integral inequality constraints can be handled within the framework of the problem given by Relationships (2.1) through (2.6).

In the optimization problem given by (2.1) through (2.6), any solution of Equation (2.2) which satisfies the Inequalities (2.3) and (2.4) and the Initial Conditions (2.6) is called a trajectory. Along any trajectory it is assumed that all functions possess derivatives of any order that may be required in the analysis.

The inequality constraints restrict the possible solutions of Equation (2.2) to regions of the (X,U,t)-space defined by Inequalities (2.3) and (2.4). The boundaries of these regions (constraint boundaries) are the surfaces $S_j(X,t) = 0$ and $C_i(X,U,t) = 0$. The point $[t_1,X(t_1)]$ where a trajectory enters a constraint boundary is known as an entering-corner point. The point where a trajectory leaves a constraint boundary, $[t_2,X(t_2)]$, is an exiting-corner point.

Many possible solutions to the differential equations, Equation (2.2), with the initial conditions, Equation

(2.6), can be obtained by choosing different functions for U. Suppose one has a solution which satisfies every inequality constraint of the form (2.3) and (2.4), except C_k and S_j. Thus,

$$C_i(X,U,t) \leq 0 \quad i \neq k, \; 1 \leq i \leq \alpha$$

$$S_i(X,t) \leq 0 \quad i \neq j, \; 1 \leq i \leq \beta$$

and

$$C_k(X,U,t) > 0$$

$$S_j(X,t) > 0$$

for some values of t in the interval, $t_0 \leq t \leq t_N$. How should the control U be chosen in order to have $C_k \leq 0$ and $S_j \leq 0$? The question is easily resolved for C_k. Suppose that $C_k(X,U,t) > 0$ at time t. Since C_k explicitly contains the control, one component of U can be found to satisfy the equation

$$C_k \; (X,U,t) = 0 \; .$$

Therefore, an inequality constraint which explicitly contains the control readily provides a means of calculating U so that a constraint boundary ($C_i = 0$) will not be crossed. For a state-variable constraint such as $S_j(X,t)$, which does not contain U, a different procedure is required. (Two such procedures are given in Section 2.1.) Thus far, only a single constraint has been zero at a time. The problem of having more than one inequality constraint simultaneously zero is discussed in Section 3.3.

2.1 State-variable inequality constraints.

The choice of U so that the constraint boundary $S_i(X,t) = 0$ will not be crossed by a solution of Equation (2.2) must now be resolved. On an arbitrary interval, $t_1 \le t \le t_2$, assume that $S_i(X,t) \ge 0$ for some solution of Equation (2.2). By the statement of the problem at the beginning of this chapter, only those solutions which remain within the region $S_i < 0$ or travel along the boundary $S_i = 0$ are sought. Hence, on the interval $t_1 \le t \le t_2$, it is required that $S_i(X,t) = 0$ in order for the solution to be admissible. If S_i is zero on $t_1 \le t \le t_2$ then every derivative of S_i must be zero, also; *i.e.*,

$$\frac{d^j S_i}{dt^j} = 0 \quad (j = 1,2,\ldots) \ . \tag{2.9}$$

Equation (2.9) may be rewritten as

$$\frac{d^j S_i}{dt^j} = \frac{\partial}{\partial X}\left[\left(\frac{d^{j-1}S_i}{dt^{j-1}}\right)\right] F(X,U,t) + \frac{\partial}{\partial t}\left(\frac{d^{j-1}S_i}{dt^{j-1}}\right) \tag{2.10}$$

for $j = 1,2,\ldots$ with

$$\frac{d^0 S_i}{dt^0} = S_i \quad \text{and} \quad \frac{dX}{dt} = F(X,U,t) \ .$$

The state-variable constraint S_i is called a q-th order constraint whenever

$$\frac{\partial}{\partial U}\left[\frac{d^q S_i}{dt^q}\right] \not\equiv 0$$

and

$$\frac{\partial}{\partial U}\left[\frac{d^j S_i}{dt^j}\right] \equiv 0 \ , \qquad (j = 0,1,\ldots, q-1) \ .$$

Hence the q-th derivative of S_i is the first one that explicitly contains the control U. The case in which $q = 1$ is treated by Berkovitz [3] and by Pontryagin et al. [43]. Dreyfus [18] and Bryson et al. [10] discuss cases in which $q>1$.

If S_i is a q-th order constraint, then on the interval $t_1 \le t \le t_2$, where $S_i = 0$, the control may be determined from

$$\left(\frac{d^q S_i}{dt^q}\right)_t = 0 \ . \tag{2.11}$$

Equation (2.11) is the required rule for choosing U on a state-variable constraint boundary.

Equation (2.11) involves X,U and t. This suggests that a state-variable inequality constraint, such as described by Inequality (2.4), may be replaced by a constraint of the form of Inequality (2.3) and some auxiliary conditions. The constraint can be written as

$$C(X,U,t) = \begin{cases} -1 \; ; & \text{if} \quad S_i(X,t)<0 \\ \dfrac{d^q S_i}{dt^q} \; ; & \text{if} \quad S_i(X,t) = 0 \end{cases} \tag{2.12}$$

so that $C(X,U,t) \leq 0$ for $t_0 \leq t \leq t_N$. The auxiliary conditions are readily obtained if one notes that in order for S_i to remain zero on the interval $t_1 \leq t \leq t_2$, when the control is given by Equation (2.11), the following conditions must be satisfied:

$$\left.\begin{aligned} (S_i)_{t_1} &= 0 \\ \left(\frac{dS_i}{dt}\right)_{t_1} &= 0 \\ &\vdots \\ \left(\frac{d^{q-1}S_i}{dt^{q-1}}\right)_{t_1} &= 0 \; . \end{aligned}\right\} \tag{2.13}$$

Equations (2.13) follow from Equation (2.9) and

$$\left(\frac{d^{j-1}S_i}{dt^{j-1}}\right)_\sigma = \left(\frac{d^{j-1}S_i}{dt^{j-1}}\right)_{t_1} + \int_{t_1}^{\sigma} \left(\frac{d^j S_i}{dt^j}\right)_t dt \ . \tag{2.14}$$

Equation (2.14) holds for $j = 1,2,\ldots, q$ and $t_1 \le \sigma \le t_2$.

If $S_i < 0$ for $t = t_2 + \varepsilon$, where ε is an arbitrary small positive number, then $d^j S_i/dt^j$ is free (for each j) at $t = t_2 + \varepsilon$. Therefore, at $t = t_2$

$$\left(\frac{d^{q-1}S_i}{dt^{q-1}}\right)_{t_2} = 0 \ . \tag{2.15}$$

Equation (2.15) is another auxiliary condition.

Therefore, a q-th order state-variable inequality constraint, $S_i(X,t)$, can be replaced, on the interval $t_0 \le t \le t_N$, by the q point-constraints given by Equations (2.13), the point-constraint given by Equation (2.15), and an inequality constraint given by Equation (2.12). The point-constraints constitute intermediate boundary conditions. (In the next chapter a new optimization problem is defined with intermediate boundary conditions and inequality constraints similar to Inequality (2.3). This problem is then investigated to obtain conditions which its solution must satisfy.)

There are two alternate procedures for dealing with a q-th order state-variable inequality constraint.

1. On the constraint boundary, the auxiliary conditions are taken as

$$\frac{d^j S}{dt^j} = 0$$

for $j = 1,2,\ldots, q$. Denham [16] points out that this type of formulation does not offer the computational advantage of the formulation according to Equations (2.12), (2.13) and (2.15).

2. The n state variables are related by q equations on a q-th order state-variable constraint boundary, [18]. Therefore, q of the state variables can be determined in terms of the remaining $n-q$ variables. Without loss of generality, the first q state variables can be written as

$$X_i = X_i(X_k, t)$$

for $i = 1,2,\ldots, q$; $k = q+1, q+2,\ldots, n$. For the n-q independent state variables, the differential equations become

$$\dot{X}_k = \bar{F}_k(X_j, U, t) \tag{2.16}$$

for $k,j = q+1, q+2,\ldots, n$.

Therefore, on a q-th order state-variable constraint boundary, $S = 0$, the n differential equations, Equation (2.2), are replaced by Equations (2.16), with the control determined from $d^qS/dt^q = 0$.

The question of using a reduced set of differential equations is discussed by Dreyfus [18,19] and Berkovitz and Dreyfus [4]. In Reference 4 the authors show the equivalence between the use of a full set of equations and the use of a reduced set. One advantage of retaining the full set of equations is that the form of Equations (2.2) does not change on a state-variable constraint boundary.

An additional point to be discussed about a q-th order state-variable constraint is the allowable bound on q. On the boundary of a q-th order constraint, the n state variables are related by q equations similar to Equations (2.13). Hence $q \leq n$ for a properly imposed constraint. A q-th order constraint with $q>n$ must be reformulated or removed from the problem. An example of such a constraint is the constraint on one state variable, when the derivative of that state variable is a constant. Such a state variable is uncontrollable.

2.2 Entering- and exiting-corner times.

A subproblem associated with inequality constraints concerns the determination of entering- and exiting-corner times. These are the times at which a trajectory enters or leaves a constraint boundary. For the constraints

$$C(X,U,t) \leq 0$$

$$S(X,t) \leq 0$$

an entering-corner time, t_1 , is easily determined from the relationships

$$C(X(t_1),U(t_1),t_1) = 0 \text{ , [or } S(X(t_1),t_1) = 0]$$

where

$$C(X(\tau),U(\tau),\tau) < 0 \text{ [or } S(X(\tau),\tau) < 0] \text{ for } t_1-\varepsilon \leq \tau < t_1,$$

with ε an arbitrary small positive number. The determination of an exiting-corner time is not as straight forward.

The analysis of an optimization problem gives a rule for determining the optimal U (see Chapter 3). This rule allows two values of U to be computed: an "unconstrained" value, $\bar{U}$, and a "constrained" value $\tilde{U}$. Use of $\bar{U}$ may cause some of the inequality constraints to be greater than zero. If this is the case, then the "constrained" value of the control must be employed. The two possible values of the control will be used to determine an exiting-corner time.

There are two cases which must be considered.

(1) The trajectory travels along the constraint boundary for a finite time interval.

For the constraint $C(X,U,t)$, an exiting-corner time, t_2, is determined from the relationships

$$C(X(t_2),\bar{U}(t_2),t_2) = 0$$

when

$$C(X(\tau),\bar{U}(\tau),\tau) > 0$$

for

$$t_2-\varepsilon \leq \tau < t_2 \ .$$

Note that

$$C(X(t),\tilde{U}(t),t) = 0$$

for

$$t_2-\varepsilon \leq t < t_2 \ .$$

Recall that a q-th order state-variable constraint $S(X,t)$ can be replaced by the conditions

$$\left(\frac{d^j S}{dt^j}\right)_{t_1} = S^{(j)}(X(t_1),t_1) = 0$$

for $j = 0,1,2,\ldots, q-1$ on the constraint boundary $S(X,t) = 0$, where t_1 is the entering-corner time. The control is determined from

$$\left(\frac{d^q S}{dt^q}\right)_{t} = S^{(q)}(X(t),\tilde{U}(t),t) = 0 \ .$$

Therefore, an exiting-corner time is that value of time, t_2 , for which

$$S^{(q)}(X(t_2),\bar{U}(t_2),t_2) = 0$$

when

$$S^{(q)}(X(\tau),\bar{U}(\tau),\tau) > 0$$

for $t_2-\varepsilon \leq \tau < t_2$.

(2) The trajectory only touches the constraint boundary at one point.

The exiting time is t_2 , and therefore

$$C(X(t_2),\bar{U}(t_2),t_2) = 0$$

$$S^{(q)}(X(t_2),\bar{U}(t_2),t_2) = 0 \ .$$

CHAPTER 3

ANALYSIS

"Therefore let every man
now task his thought".

King Henry V
Act I Scene II

3.1 A general problem.

A general problem with inequality constraints and intermediate boundary conditions can now be stated: determine the control program $U(t)$, $t_0 \le t \le t_N$, so as to minimize the functional

$$\Gamma = G(X(t_N),t_N) + \sum_{j=1}^{N} \int_{t_{j-1}^{+}}^{t_j^{-}} Q(X,U,t)dt \tag{3.1}$$

subject to the conditions

$$\dot{X} = F(X,U,t) \tag{3.2}$$

$$C_i(X,U,t) \le 0 \quad (i = 1,2,\ldots, r) \tag{3.3}$$

on the intervals $t_{j-1}^{+} \le t \le t_j^{-}$ $(j = 1,2,\ldots, N)$, where t_0 is specified and $X(t_0)$ is known, i.e.,

$$X(t_0) = X^{\circ} \tag{3.4}$$

and where

$$L^{(j)}(X(t_j),t_j) = 0 \quad (j = 1,2,\ldots, N) \; . \tag{3.5}$$

The times t_j may be unknown. It is assumed that $t_i = t_i^+ = t_i^-$, for $i = 0,1,2,\ldots, N$. The use of the superscripts + and - is explained in the notation section of Chapter 1. A solution of Equation (3.2) which satisfies Inequalities (3.3), Equations (3.5) and the Initial Conditions (3.4) will be called an optimal trajectory. The notation used in Relationships (3.1) through (3.5) is defined as follows:

$$X = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \cdot \\ \cdot \\ \cdot \\ X_n(t) \end{bmatrix}$$

, an n-dimensional vector of state variables with each X_i continuous on the interval $t_0 \le t \le t_N$;

$$U = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \cdot \\ \cdot \\ \cdot \\ U_m(t) \end{bmatrix}$$

, an m-dimensional vector of control variables;

$$F = \begin{bmatrix} F_1(X,U,t) \\ F_2(X,U,t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(X,U,t) \end{bmatrix}$$

, a specified n-dimensional vector function with each F_i defined on the intervals $t_{j-1}^{+} \leq t \leq t_j^{-}$;

$$L^{(j)} = \begin{bmatrix} L_1^{(j)}(X(t_j),t_j) \\ L_2^{(j)}(X(t_j),t_j) \\ \cdot \\ \cdot \\ \cdot \\ L_{\ell_j}^{(j)}(X(t_j),t_j) \end{bmatrix}$$

, a specified ℓ_j-dimensional vector function of point-constraints, representing intermediate boundary conditions;

$G(X(t_N),t_N)$ and $Q(X,U,t)$, specified scalar functions; and

$$C = \begin{bmatrix} C_1(X,U,t) \\ C_2(X,U,t) \\ \cdot \\ \cdot \\ \cdot \\ C_r(X,U,t) \end{bmatrix}$$

, a specified r-dimensional vector of inequality constraint functions satisfying $\partial C_i/\partial U \not\equiv 0$ whenever $C_i = 0$, for each i.

The points $t = t_j$ $(j = 1,2,\ldots, N-1)$ denote the times at which the trajectory enters or leaves a state-variable constraint boundary (see Chapter 2), or the times at which some $F_i(X,U,t)$ has a finite jump discontinuity.

It will be assumed that all functions possess derivatives of any order which may be required in the analysis. This property is to hold on each interval $t_{j-1}^{+} \leq t \leq t_j^{-}$ $(j = 1,2,\ldots, N)$. Furthermore, if a function is discontinuous at $t = t_j$ then it is assumed that unique right and left limits exist.

It is noted that problems containing discontinuous state variables, [13], may be handled also. For example, suppose that at $t = t_1$

$$X_1(t_1^{+}) = X_1(t_1^{-}) + c \ ,$$

where c is a constant. Introduce a new state variable X_{n+1}. Then

$$\dot{X}_1 = F_1(X_1,X_2,\ldots,X_n,U,t) \ , \ X_1(t_0) \text{ given}$$

$$\dot{X}_{n+1} = 0 \ , \quad X_{n+1}(t_0) = X_1(t_1^{-}) + c$$

for $t_0 \leq t \leq t_1^{-}$, and

$$\dot{X}_1 = 0 \ , \quad X_1(t_1^{+}) = X_1(t_1^{-})$$

$$\dot{X}_{n+1} = F_1(X_{n+1},X_2,\ldots,X_n,U,t) \ , \ X_{n+1}(t_1^{+}) = X_1(t_1^{-}) + c \ .$$

Use of the extra state variable, X_{n+1}, has removed the discontinuity on the state variable X_1. Henceforth the total number of state variables, n, will be assumed to consist of the original state variables plus the extra state variables introduced to remove discontinuities.

In order to include the constraints given by Inequalities (3.3) in the analysis, the method due to Valentine [48] will be employed. Let the real number Z_k be defined by

$$Z_k^2 + C_k(X,U,t) = 0 \ . \tag{3.6}$$

As Z_k is to be a real number, $Z_k^2 \geq 0$ and thus Equation (3.6) can replace the inequality $C_k \leq 0$, for $k = 1,2,\ldots, r$. By means of Equation (3.6), the inequality constraints have been converted into equality constraints.

The classical method of unknown multipliers will be used in order to study the effects of the intermediate boundary conditions, the terminal conditions and the inequality constraints on the functional being minimized, and also to obtain the optimal choice for the control.

Adjoin Equations (3.2), (3.5) and (3.6) to Equation (3.1) by unknown Lagrange multipliers P_0, P, $\nu^{(j)}$ and M to define a new scalar function V:

$$V = \sum_{j=1}^{N} \left\{ R^{(j)} + \int_{t_{j-1}^{+}}^{t_j^{-}} (H - P^T\dot{X} + M^T Z^2)dt \right\} \tag{3.7}$$

where

$$R^{(N)} = P_0 G(X(t_N),t_N) + [\nu^{(N)}]^T L^{(N)}(X(t_N),t_N) \tag{3.8}$$

$$R^{(j)} = [\nu^{(j)}]^T L^{(j)}(X(t_j),t_j) \;, \; (j = 1,2,\ldots, N-1) \tag{3.9}$$

$$Z^2 = \begin{bmatrix} Z_1^2 \\ Z_2^2 \\ \cdot \\ \cdot \\ \cdot \\ Z_r^2 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix}, \quad \nu^{(j)} = \begin{bmatrix} \nu_1^{(j)} \\ \nu_2^{(j)} \\ \cdot \\ \cdot \\ \cdot \\ \nu_{\ell_j}^{(j)} \end{bmatrix}, \quad M = \begin{bmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \\ \cdot \\ M_r \end{bmatrix}$$

and P_0 is a scalar constant. The variational Hamiltonian, H, is defined by

$$H = P_0 Q(X,U,t) + P^T F(X,U,t) + M^T C(X,U,t) \;. \tag{3.10}$$

Minimization of Equation (3.1), subject to the requirement that Equations (3.2), (3.4), and (3.5) and Inequalities (3.3) be satisfied is equivalent to minimization of Equation (3.7) with initial conditions given by Equation (3.4) as a side condition.

Assume that a minimizing trajectory exists; then Equation (3.7) will be expanded in a Taylor series about this trajectory. The first order terms in the Taylor series constitute the first variation of V, denoted by $\delta'V$; the second order terms, except for a factor of $\frac{1}{2}$, compose the second variation of V, $\delta''V$. In the following sections both the first and the second variations of V will be obtained. By evaluating $\delta'V$ and $\delta''V$ on a minimizing trajectory, the conditions which the trajectory must satisfy will be obtained. For a solution to be a minimizing trajectory it is required that $\delta' V = 0$ and $\delta'' V \geq 0$ for arbitrary small variations about the solution, [5].

Let a candidate trajectory give a value of V as

$$V = V_0(X,P,U,M,Z^2,\nu^{(j)},t_j) .$$

A nearby trajectory will give

$$V = V_1(X+\Delta X,P+\Delta P,U+\Delta U,M+\Delta M,Z^2+\Delta Z^2,\nu^{(j)}+\Delta\nu^{(j)},t_j+\Delta t_j) .$$

Then the change in V is given by

$$\Delta V = V_1 - V_0 = (\delta' V)_{V_0} + \frac{1}{2!}(\delta'' V)_{V_0} + \ldots .$$

3.2 The first variation.

The first order terms in the Taylor series expansion of V are

$$\delta' V = \sum_{j=1}^{N} \left\{ \Delta R^{(j)} + \Delta \int_{t_{j-1}^{+}}^{t_j^{-}} (H - P^T\dot{X} + M^T Z^2)dt \right\} \qquad (3.11)$$

Expansion of the terms outside the integral sign gives

$$\Delta R^{(j)} = \left[R_X^{(j)}\Delta X + R_t^{(j)}\Delta t + R_\nu^{(j)}\Delta\nu^{(j)} \right]_{t_j} \qquad (3.12)$$

for $j = 1,2,\ldots, N-1$, and

$$\Delta R^{(N)} = \left[R_X^{(N)} \Delta X + R_t^{(N)} \Delta t + R_\nu^{(N)} \Delta \nu^{(N)} + R_{P_0}^{(N)} \Delta P_0 \right]_{t_N} . \qquad (3.13)$$

The subscript notation R_X denotes partial differentiation, while $(R)_t$ denotes the value of R at t. The superscript (j) on $\nu^{(j)}$ has been dropped whenever partial differentiation with respect to $\nu^{(j)}$ is required.

By Leibnitz's Rule, the variation of each of the integral terms is

$$\Delta \int_{t_{j-1}^+}^{t_j^-} (H-P^T\dot{X}+M^T Z^2)dt = [(H-P^T\dot{X}+M^T Z^2)\Delta t]_{t_j^-}$$
$$- [(H-P^T\dot{X}+M^T Z^2)\Delta t]_{t_{j-1}^+}$$
$$+ \int_{t_{j-1}^+}^{t_j^-} \delta(H-P^T\dot{X}+M^T Z^2)dt . \qquad (3.14)$$

When expanded, the last term in Equation (3.14) is

$$\int_{t_{j-1}^+}^{t_j^-} \delta(H-P^T\dot{X}+M^T Z^2)dt =$$
$$\int_{t_{j-1}^+}^{t_j^-} (H_X x+H_P p+H_U u+H_M \mu-\dot{X}^T p-P^T \dot{x}+(Z^2)^T \mu+M^T \delta(Z^2)+H_{P_0} p_0)dt \qquad (3.15)$$

where

$$x = \delta X, \ \mu = \delta M, \ u = \delta U, \ p = \delta P$$

and

$$\delta\dot{X} = \frac{d}{dt}(\delta X) = \dot{x} .$$

Now, since

$$M^T Z^2 = \sum_{i=1}^{r} M_i Z_i^2$$

it follows that

$$M^T \delta(Z^2) = 2 \sum_{i=1}^{r} M_i Z_i z_i \tag{3.16}$$

where $z_i = \delta Z_i$. If the term $-P^T \dot{x}$ under the integral in Equation (3.15) is integrated by parts, and if the relation

$$\Delta X = x + \dot{X}\,\Delta t \tag{3.17}$$

is used to combine terms, the first variation of V can be obtained in the form of Equation (3.18). Note that $\Delta t_0 = 0$ and $x(t_0) = 0$ by Equation (3.4) since $X(t_0)$ is known, and that

$$\Delta X(t_j^+) = \Delta X(t_j^-) = \Delta X(t_j)$$

since X is assumed continuous at t_j.

$$\delta'V = [(R_X^{(N)}-P^T)\Delta X + (R_t^{(N)}+H+M^T Z^2)\Delta t + R_\nu^{(N)}\Delta\nu^{(N)} + R_{P_0}^{(N)}\Delta P_0]_{t_N}$$

$$+ \sum_{j=1}^{N-1} R_\nu^{(j)}(t_j)\Delta\nu^{(j)}$$

$$+ \sum_{j=1}^{N-1} [R_X^{(j)}(t_j)-P^T(t_j^-)+P^T(t_j^+)]\Delta X(t_j)$$

$$+ \sum_{j=1}^{N-1} [R_t^{(j)}(t_j)+H(t_j^-)-H(t_j^+)+M^T(t_j^-)Z^2(t_j^-)-M^T(t_j^+)Z^2(t_j^+)]\Delta t_j$$

$$+ \sum_{j=1}^{N} \int_{t_{j-1}^+}^{t_j^-} [(H_X+\dot{P}^T)x+(H_P-\dot{X}^T)p+H_U u$$

$$+(H_M+(Z^2)^T)\mu+H_{P_0}p_0+2\sum_{i=1}^{r} M_i Z_i z_i]dt \qquad (3.18)$$

If the trajectory makes V an extremum then $\delta'V = 0$. The consequences of this are examined in the next section.

3.3 Conditions obtained from the first variation.

The first variation of $V, \delta'V$, must be zero on a minimizing trajectory, [5]. Requiring that Equation (3.18) be zero leads to the following conditions.

(1) At t_N.

$$[P^T(t_N) - R_X^{(N)}(t_N)]\Delta X(t_N) = 0 \tag{3.19}$$

$$[R_t^{(N)}(t_N) + H(t_N) + M^T(t_N)Z^2(t_N)]\Delta t_N = 0 \tag{3.20}$$

$$[R_\nu^{(N)}(t_N)]\Delta\nu^{(N)} = 0 \tag{3.21}$$

$$[R_{P_0}^{(N)}(t_N)]\Delta P_0 = 0 \tag{3.22}$$

(2) At t_j , $(j = 1,2,\ldots, N-1)$.

$$[R_X^{(j)}(t_j) - P^T(t_j^-) + P^T(t_j^+)]\Delta X(t_j) = 0 \tag{3.23}$$

$$[R_t^{(j)}(t_j)+H(t_j^-)-H(t_j^+)+M^T(t_j^-)Z^2(t_j^-)-M^T(t_j^+)Z^2(t_j^+)]\Delta t_j = 0 \tag{3.24}$$

$$[R_\nu^{(j)}(t_j)]\Delta\nu^{(j)} = 0 \tag{3.25}$$

In Equations (3.19), (3.23), (3.21) and (3.25) the quantities in the brackets are zero on the optimal trajectory. In Equations (3.20) and (3.24) either t_i is known $(\Delta t_i = 0)$ or the quantity in the brackets is zero.

Equations (3.19) through (3.22) are the terminal conditions. Equations (3.23) through (3.25) are the corner conditions. Note that P and H may be discontinuous at t_j , $(j = 1,2,\ldots, N-1)$.

(3) For $t_{j-1}^{+} \leq t \leq t_{j}^{-}$, $(j = 1,2,\ldots, N)$, the Euler-Lagrange equations are given by

$$\dot{X} = H_P^T \tag{3.26}$$

$$\dot{P} = -H_X^T \tag{3.27}$$

$$0 = H_U^T \tag{3.28}$$

$$0 = H_M^T + Z^2 \tag{3.29}$$

$$0 = H_{P_0} p_0 \tag{3.30}$$

$$0 = M_i Z_i \quad , \quad (i = 1,2,\ldots, r) \ . \tag{3.31}$$

Equation (3.26) is merely Equation (3.2), and Equation (3.29) is Equation (3.6) written in vector form. Since P_0 is a constant, $p_0 = \Delta P_0$. Furthermore, neither $H_{P_0} = Q$ nor $R_{P_0}^{(N)}(t_N) = G$ are both identically zero, hence $p_0 = 0$. But Equation (3.1) is to be minimized; therefore $P_0 \geq 0$. Problems in which P_0 is zero are called abnormal and will not be considered here. Since P_0 is an arbitrary positive constant, it can be set equal to unity: henceforth, $P_0 = 1$ will be used.

Equation (3.31) can be combined with Equation (3.6) to give

$$M_i C_i = 0 \quad , \quad (i = 1,2,\ldots, r) \; .$$

The differential equations which the optimal trajectory must satisfy are restated as

$$\left.\begin{aligned} \dot{X} &= F(X,U,t) \\ \dot{P} &= -Q_X^T - F_X^T P - C_X^T M \end{aligned}\right\} \qquad (3.32)$$

where M and U are determined from

$$\left.\begin{aligned} 0 &= M_i C_i(X,U,t), \quad (i = 1,2,\ldots, r) \\ 0 &= Q_U^T + F_U^T P + C_U^T M \; . \end{aligned}\right\} \qquad (3.33)$$

Equations (3.33) are $r + m$ algebraic (and usually nonlinear) equations in the $r + m$ unknowns M_i $(i = 1,2,\ldots, r)$ and U_k $(k = 1,2,\ldots, m)$ in terms of X,P and t. Now, if $C_i < 0$ then $M_i = 0$; if $C_i = 0$ then M_i must be determined. The maximum number of constraints C_i that can be zero at any given time must be ascertained.

Suppose that r_1 $(r_1 \leq r)$ of the Inequality Constraints (3.3) are simultaneously zero at time t. Form these r_1 constraints into a vector E. The $r_1 \times m$ matrix E_U must be of rank r_1 $(r_1 \leq m)$ in order to solve for r_1 of the controls in terms of the remaining $m-r_1$ controls, the state X, and the time t, [1], in the equation $E(X,U,t) = 0$. Therefore, the maximum number of constraints which can simultaneously be zero must be less than or equal to the number of control variables. The r_1 constraints must be independent in the sense that the $r_1 \times m$ matrix E_U is required to be of full rank.

Let the vector W contain those M_i corresponding to a constraint that is zero. The k-th component of $C_X^T M$ is

$$\sum_{i=1}^{r} \frac{\partial C_i}{\partial X_k} M_i = \sum_{j=1}^{r_1} \frac{\partial E_j}{\partial X_k} W_j ,$$

because $M_i = 0$ whenever $C_i < 0$. Therefore,

$$C_X^T M = E_X^T W$$

and similarly,

$$C_U^T M = E_U^T W .$$

To obtain information on the possible finite jump discontinuities in P, the second equation in each of Equations (3.32) and (3.33) is rewritten as

$$\dot{P} = -Q_X^T - F_X^T P - E_X^T W \qquad (3.34)$$

and

$$0 = Q_U^T + F_U^T P + E_U^T W \ . \tag{3.35}$$

Since the $r_1 \times m$ matrix E_U has rank r_1, there is a nonsingular $r_1 \times r_1$ submatrix, $\bar{E}_U$, of E_U, such that Equation (3.35) can be written as the two following equations:

$$\bar{Q}_U^T + \bar{F}_U^T P + \bar{E}_U^T W = 0$$

$$\tilde{Q}_U^T + \tilde{F}_U^T P + \tilde{E}_U^T W = 0 \ .$$

The first of these vector equations represents r_1 algebraic equations; the second, $m-r_1$ algebraic equations. Solving for W gives

$$W = -(\bar{E}_U^T)^{-1}[\bar{Q}_U^T + \bar{F}_U^T P] \ , \tag{3.36}$$

since $\bar{E}_U$ is nonsingular. On substituting W into Equation (3.34);

$$0 = \dot{P} + [Q_X^T - E_X^T(\bar{E}_U^T)^{-1}\bar{Q}_U^T] + [F_X^T - E_X^T(\bar{E}_U^T)^{-1}\bar{F}_U^T]P \ . \tag{3.37}$$

Let $S(X,t)$ be a q-th order state-variable constraint. Assume that the trajectory enters the constraint boundary at $t = t_1$ and leaves at $t = t_2$. For simplicity, assume only one constraint will be zero during this time interval, therefore $r_1 = 1$. From Equations (2.13) and (2.15), it follows that

$$L_k^{(1)}(X(t_1),t_1) = \left(\frac{d^{k-1}S}{dt^{k-1}}\right)_{t_1} , \quad (k = 1,2,\ldots, q)$$

and

$$L^{(2)}(X(t_2),t_2) = \left(\frac{d^{q-1}S}{dt^{q-1}}\right)_{t_2} .$$

From Equation (3.23):

$$\left.\begin{aligned} P^T(t_1^+) &= P^T(t_1^-) - \sum_{j=1}^{q} \nu_j^{(1)} \left[\frac{\partial}{\partial X}\left(\frac{d^{j-1}S}{dt^{j-1}}\right)\right]_{t_1} \\ P^T(t_2^+) &= P^T(t_2^-) - \nu^{(2)} \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right]_{t_2} \end{aligned}\right\} \qquad (3.38)$$

By Equations (3.38), the Lagrange multipliers, P_k, may be discontinuous at both t_1 and t_2. Following an argument given by Bryson <u>et al</u>. [10] for the case of one control variable (m=1), it will be shown that P can be continuous at $t = t_2$. In the expression on the right hand side of Equation (3.37), replace P by

$$P + \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right]^T b$$

where b is an arbitrary scalar constant. The new expression is

$$\frac{d}{dt}\left\{ P + \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right]^T b \right\} + [Q_X^T - E_X^T(\bar{E}_U^T)^{-1}\bar{Q}_U^T]$$

$$+ [F_X^T - E_X^T(\bar{E}_U^T)^{-1}\bar{F}_U^T] \left\{ P + \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right]^T b \right\} .$$

Subtracting Equation (3.37) from this expression gives

$$\frac{d}{dt}\left\{ \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right]^T b \right\} + [F_X^T - E_X^T(\bar{E}_U^T)^{-1}\bar{F}_U^T] \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right]^T b \, . \qquad (3.39)$$

Now, on a state-variable constraint boundary a component of the control vector is determined from the equation

$$0 = E = \frac{d^qS}{dt^q} = \frac{d}{dt}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) = \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right] F + \frac{\partial}{\partial t}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) .$$

Thus,

$$E_U = \frac{\partial}{\partial U}\left(\frac{d^qS}{dt^q}\right) = \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right] F_U$$

since the $q-1$ derivative of S does not contain U.

Now

$$\frac{d}{dt}\left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right] = \left[\frac{\partial}{\partial X}\left\{ \frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right\} \right] F$$

$$+ \frac{\partial}{\partial t}\left\{ \frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right) \right\}$$

and therefore

$$\frac{d}{dt}\left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right] = E_X - \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right] F_X .$$

Since b is a constant, Expression (3.39) reduces to

$$\left\{E_X^T - F_X^T\left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right]^T\right\} b$$
$$+[F_X^T - E_X^T(\bar{E}_U^T)^{-1}\bar{F}_U^T]\left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right]^T b .$$

Using the fact that

$$\bar{E}_U = \left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right]\bar{F}_U$$

this expression becomes

$$\{E_X^T - E_X^T\,(\bar{E}_U^T)^{-1}(\bar{E}_U^T)\}\ b = 0 .$$

Hence Equation (3.37) is unchanged whenever

$$\left[\frac{\partial}{\partial X}\left(\frac{d^{q-1}S}{dt^{q-1}}\right)\right]^T b$$

is added to P. If this addition is made at each t in the interval $t_1^+ \leq t \leq t_2^-$, then at $t = t_2$, $P(t_2^+)$ is given by Equation (3.38):

$$P^T(t_2^+) = P^T(t_2^-) + (b-\nu^{(2)}) \left[\frac{\partial}{\partial X} \left(\frac{d^{q-1}S}{dt^{q-1}} \right) \right]_{t_2} .$$

If $b = \nu^{(2)}$, then $P(t_2^+) = P(t_2^-)$. Hence, P can be made continuous at points where the trajectory leaves a state-constraint boundary [21] by proper choice of the arbitrary constant b; but P may be discontinuous at points where the trajectory enters a state-constraint boundary. The preceding argument can be extended to cases in which more than one state-variable constraint is simultaneously zero. The expression used to replace P in Equation (3.37) is

$$P + \left\{ \frac{\partial}{\partial X} \begin{bmatrix} S_1^{(1)} \\ S_2^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ S_\alpha^{(\alpha)} \end{bmatrix} \right\}^T b$$

where

$$S_j^{(j)} = \frac{d^{q_j-1} S_j}{dt^{q_j-1}} , \quad (j = 1,2,\ldots, \alpha)$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_\alpha \end{bmatrix} .$$

A final consequence of $\delta' V = 0$ is related to Equation (3.28). Since this equation is written in vector notation, it actually represents m algebraic equations. If any one of these equations were identically zero, independent of some component of U, there is a singular extremal. This will usually occur for $C_i < 0$. Kopp and Moyer [34] give conditions to determine U for singular extremals.

3.4 The second variation.

The second order terms in the Taylor series expansion of

$$V = V_i(X+\Delta X, P+\Delta P, U+\Delta U, M+\Delta M, Z^2+\Delta Z^2, \nu^{(j)}+\Delta\nu^{(j)}, t_j+\Delta t_j)$$

about $(X,P,U,M,Z^2,\nu^{(j)},t_j)$ constitute the second variation of V, $\delta'' V$, except for a factor of $\frac{1}{2}$. In particular, an optimum trajectory is required to have $\delta' V = 0$, $\delta'' V \geq 0$. The second variation of V is given below for a trajectory which satisfies $\delta' V = 0$.

$$\begin{aligned}
\delta''V = {} & [\Delta X(t_N)]^T[R_{XX}^{(N)}\Delta X + R_{X\nu}^{(N)}\Delta\nu^{(N)} + R_{Xt}^{(N)}\Delta t - \Delta P]_{t_N} \\
& + [\Delta\nu^{(N)}]^T[R_{\nu X}^{(N)}\Delta X + R_{\nu t}^{(N)}\Delta t]_{t_N} \\
& + [\Delta t_N][(R_{tX}^{(N)} + H_X)\Delta X + (R_{tt}^{(N)} + H_t)\Delta t \\
& \qquad + H_P\Delta P + R_{t\nu}^{(N)}\Delta\nu^{(N)}]_{t_N} \\
& + \sum_{j=1}^{N-1}[\Delta\nu^{(j)}]^T[R_{\nu X}^{(j)}\Delta X + R_{\nu t}^{(j)}\Delta t]_{t_j} \\
& + \sum_{j=1}^{N-1}[\Delta X(t_j)]^T[R_{XX}^{(j)}\Delta X + R_{X\nu}^{(j)}\Delta\nu^{(j)} + R_{Xt}^{(j)}\Delta t \\
& \qquad -\Delta P(t_j^-) + \Delta P(t_j^+)]_{t_j} \\
& + \sum_{j=1}^{N-1}[\Delta t_j][(R_{tt}^{(j)} + H_t(t_j^-) - H_t(t_j^+))\Delta t_j + R_{t\nu}^{(j)}\Delta\nu^{(j)} \\
& \qquad + H_P(t_j^-)\Delta P(t_j^-) - H_P(t_j^+)\Delta P(t_j^+) \\
& \qquad + (R_{tX}^{(j)} + H_X(t_j^-) - H_X(t_j^+))\Delta X]_{t_j} \\
& + \sum_{j=1}^{N}\int_{t_{j-1}^+}^{t_j^-}[x^T H_{XX}x + 2x^T H_{XP}p + 2x^T H_{XU}u + x^T\dot{p} \\
& \qquad + 2p^T H_{PU}u - p^T\dot{x} + u^T H_{UU}u \\
& \qquad + 2u^T C_U^T\mu + 2x^T C_X^T\mu + 2\sum_{i=1}^{r}(M_i z_i + 2\mu_i Z_i)z_i]dt \qquad (3.40)
\end{aligned}$$

where the second partial derivatives of

$$H = Q + P^T F + M^T C = \bar{H} + M^T C$$

are given by

$$H_{XX} = \bar{H}_{XX} + \frac{\partial}{\partial X}(M^T C_X)$$

$$H_{XP} = \bar{H}_{XP} = (H_{PX})^T$$

$$H_{XU} = \bar{H}_{XU} + \frac{\partial}{\partial U}(M^T C_X) = (H_{UX})^T$$

$$H_{UU} = \bar{H}_{UU} + \frac{\partial}{\partial U}(M^T C_U)$$

$$H_{PU} = \bar{H}_{PU} = (H_{UP})^T \ .$$

Both the matrices H_{XX} and H_{UU} are symmetric. The terms

$$2u^T C_U^T \mu + 2x^T C_X^T \mu + 2\sum_{i=1}^{r} (M_i z_i + 2\mu_i Z_i) z_i$$

under the integral in Equation (3.40) vanish, since $M_i Z_i = 0$ implies $M_i C_i = 0$, and therefore

$$\delta(M_i Z_i) = \mu_i Z_i + M_i z_i = 0$$

$$\delta(M_i C_i) = \mu_i C_i + M_i(\frac{\partial C_i}{\partial X}x + \frac{\partial C_i}{\partial U}u) = 0$$

so that

(1) $Z_i \neq 0$ $(C_i<0)$ then $M_i \equiv 0$ and $\mu_i = 0$

(2) $Z_i = 0$ $(C_i=0)$ then $M_i \not\equiv 0$ and $z_i = 0$,

$$\frac{\partial C_i}{\partial X}x + \frac{\partial C_i}{\partial U}u = 0 \ .$$

If $\dot{x}, x, u, p$ and $\dot{p}$ are "sufficiently small" then the total change in V will be given by $\Delta V = \delta' V + \frac{1}{2}\delta'' V$. Since $\delta' V$ is required to be zero, for an extremal, it follows that $\Delta V = \frac{1}{2}\delta'' V$. In order that the trajectory afford a local minimum to V, ΔV must be nonnegative for all nearby trajectories; i.e., $\Delta V \geq 0$ for all "sufficiently small" variations x and u whenever $p = \dot{p} = 0$. The effect of this requirement is now examined.

3.5 Conditions obtained from the second variation.

For the minimizing trajectory,

$$\Delta V = \frac{1}{2}\delta'' V \geq 0 \ . \tag{3.41}$$

In Equation (3.40) set $\Delta X(t_j)$, $\Delta\nu^{(j)}$, Δt_j, p and $\dot{p}$ equal to zero. Then ΔV is given by

$$\Delta V = \frac{1}{2}\sum_{j=1}^{N}\int_{t_{j-1}^{+}}^{t_{j}^{-}}[x^T H_{XX}x + 2x^T H_{XU}u + u^T H_{UU}u]dt \ .$$

The expression for ΔV can be rewritten as

$$\Delta V = \frac{1}{2}\sum_{j=1}^{N}\int_{t_{j-1}^{+}}^{t_{j}^{-}}\omega(x,u,t)dt \tag{3.42}$$

where

$$\omega(x,u,t) = [x^T \ u^T]\begin{bmatrix} H_{XX} & H_{XU} \\ H_{UX} & H_{UU}\end{bmatrix}\begin{bmatrix} x \\ u\end{bmatrix} .$$

ΔV is nonnegative if the matrix

$$\begin{bmatrix} H_{XX} & H_{XU} \\ H_{UX} & H_{UU}\end{bmatrix}$$

is positive semi-definite or positive definite. It is clear that u can be so chosen that the term $u^T H_{UU}u$ will dominate the others, i.e., $\|x\|$ will be small. Therefore it is necessary that H_{UU} be positive semi-definite or positive definite, in order to have ΔV nonnegative. The latter requirement on H_{UU}, the strengthened Legendre Condition of the Calculus of Variations, can be expressed as

$$u^T H_{UU} u > 0 \tag{3.43}$$

for arbitrary, nonzero u. Henceforth, it is assumed that Inequality (3.43) holds. The matrix

$$K = \begin{bmatrix} H_{XX} & H_{XU} \\ & \\ H_{UX} & H_{UU} \end{bmatrix}$$

may not be positive semi-definite, but ΔV could still be nonnegative (see Section 4.4). If K is positive semi-definite then the matrix $H_{XX} - H_{XU}H_{UU}^{-1}H_{UX}$ is also positive semi-definite, [27], as is the matrix H_{XX}.

Equation (3.28) combined with Inequality (3.43) shows that on an optimal trajectory the scalar function H is minimized with respect to the control U.

A stronger condition than that expressed by Equation (3.28) and Inequality (3.43) is the Weierstrass Condition

$$H(X,P,U,M,t) \le H(X,P,U^*,M,t) \tag{3.44}$$

with $M = 0$ and where U^* is any admissible control which satisfies the Inequalities (3.3), [23].

Consider now the question of the range for M_i $(i = 1,2,\ldots, r)$. Suppose that U_j can be determined from the constraint C_i. Let U_j^* denote the unconstrained optimum value of U_j; let $\bar{U}_j$ denote the constrained optimum value of U_j; then

$$C_i(X,U_j^*,t) \geq C_i(X,\bar{U}_j,t) = 0 \ . \tag{3.45}$$

There are two possible cases: $U_j^* \geq \bar{U}_j$ and $U_j^* \leq \bar{U}_j$.

(1) $U_j^* \geq \bar{U}_j$.

If the constraint were not present then $Q + P^TF$ is minimized with respect to U, and therefore

$$\left[\frac{\partial}{\partial U_j}(Q + P^TF)\right]_{\bar{U}_j} \leq \left[\frac{\partial}{\partial U_j}(Q + P^TF)\right]_{U_j^*} = 0 \ ,$$

since $Q + P^TF$ is decreasing at $\bar{U}_j$ as U_j increases. With a constraint in the problem, the term M_iC_i must be added to $Q + P^TF$, so that $\frac{\partial}{\partial U}(Q + P^TF + M_iC_i) = 0$. Thus,

$$M_i\left(\frac{\partial C_i}{\partial U_j}\right)_{\bar{U}_j} \geq 0 \ .$$

Now $(\partial C_i/\partial U_j) \geq 0$ at $U_j = \bar{U}_j$, by Inequality (3.45), and therefore $M_i \geq 0$.

(2) $U_j^* \leq \bar{U}_j$.

A similar argument, utilizing the fact that $Q + P^TF$ is decreasing at $\bar{U}_j$ as U_j decreases, shows that $M_i \geq 0$. Then, for each $i = 1,2,\ldots, r$

$$M_i(t) \geq 0 \tag{3.46}$$

on the intervals $t^+_{j-1} \leq t \leq t^-_j$, $(j = 1,2,\ldots, N)$.

3.6 Conjugate points.

An important condition that must be satisfied by the trajectory is the absence of any conjugate points. If the trajectory contained a conjugate point, then the trajectory is not optimal. There are two possible types of conjugate points; points conjugate to the initial time t_0 and points conjugate to the terminal time t_N. Breakwell and Ho [7] discuss a procedure for determining the existence of points conjugate to the terminal time, for problems without inequality constraints. In the procedure, the determinant of a certain matrix is examined at each point on the trajectory. The determinant is zero at t_N. There is a conjugate point at τ, $t_0 \leq \tau < t_N$, if the determinant is zero at τ, but has a nonzero value for t, $\tau < t < t_N$. It is conjectured that this procedure, with modifications to handle the inequality constraints, is applicable to inequality-constrained problems. In Section 4.4 tests for the existence of points conjugate to t_0 and t_N are discussed for inequality-constrained problems.

In the preceding sections of this chapter, conditions have been found which the trajectory must satisfy in order to minimize the functional given in Equation (3.1). These conditions do not indicate a procedure to generate such a trajectory. Hence, a major engineering and mathematical problem is left unresolved. This problem is discussed in Chapter 4.

CHAPTER 4

COMPUTATIONAL ALGORITHMS

"Bloody instructions,
which being taught,
return to plague the
inventor."

MacBeth
Act I Scene VII

The construction of an optimal trajectory is generally a difficult problem. The functions $X(t)$, $P(t)$, $U(t)$ and $M(t)$ must be found which satisfy

1. the differential equations; Equations (3.32):
2. the optimality conditions; Equations (3.33), Inequality (3.43) [or Inequality (3.44)]:
3. the boundary conditions; Initial Conditions (3.4) and Equations (3.19):
4. the terminal constraints; Equations (3.20) and (3.21): and
5. the corner conditions; Equations (3.23), (3.24) and (3.25).

In addition, the trajectory must not contain a conjugate point.

To satisfy all of these criteria implies that at least a two-point boundary value problem (generally nonlinear) must be solved. The boundary conditions are split between the Initial Conditions (3.4) on X, at t_0, and the terminal conditions on P, Equations (3.19), at t_N. Equations (3.25) are intermediate boundary conditions.

Methods which have been proposed for solving the split boundary value problems arising in optimization theory include (1) the gradient (steepest-ascent) methods [14,17,30,32,47], (2) quasilinearization (generalized Newton-Raphson) methods [39,40], (3) dynamic programming [18,19], (4) nonlinear programming [25,26], and (5) perturbation methods [8,29,31,33]. The penalty function technique [28,30,42,44] may be successfully employed in the solution of inequality-constrained problems. At present there is no universal algorithm which will solve all optimization problems. Some of the preceding methods (the gradient methods) will converge to a good approximation to the optimal trajectory, starting from crude initial estimates. The gradient methods may give a trajectory which does not satisfy Equations (3.33) and Inequality (3.43). Others, such as the perturbation methods can converge to the optimal trajectory, but often require good initial estimates. The best procedure may require two different methods to generate the optimum trajectory. The initial method gives a good approximation to the solution when starting from crude initial estimates. The second method, which uses the answers from the first method as its initial conditions, is used to obtain convergence to the final answer. Only one particular method, a perturbation method, will be considered here. This perturbation procedure is an iterative, rapidly converging method (provided the initial estimates are sufficiently accurate). Each iterant satisfies the Initial Conditions (3.4), the Differential Equations (3.32), and the optimality conditions, Equations (3.33) and Inequality (3.43). The method would be extremely useful for generating a sequence of optimum trajectories, each with a slightly different set of Initial Conditions (3.4), starting from a known optimum trajectory.

It will be assumed that a problem of interest has been formulated and then analyzed to obtain the information required by the analysis presented in Chapter 3. The differential equations governing X and P are known explicitly, as are the equations giving U in terms of X, P and t. All that remains is to produce a trajectory, and to test for various properties which could not be accounted for until the trajectory has been obtained (such as conjugate points).

The procedure for generating a trajectory will involve guessing (or obtaining from another method) initial estimates for $P(t_0)$, t_j, and the unknown multipliers $\nu^{(j)}$. Once these values are known, a constrained trajectory can be obtained. This trajectory will satisfy all the required conditions except perhaps the boundary conditions (intermediate and terminal) and the corner conditions. The question to be considered is the following. How should $P(t_0)$, t_j and $\nu^{(j)}$ be changed so that the boundary and corner conditions will be better satisfied? The perturbation method presented here answers this question and thereby provides an algorithm which produces the optimal constrained trajectory.

Equation (3.23) shows that the Lagrange multipliers, P_k, may have finite jump discontinuities at points where the trajectory enters a state-variable constraint boundary. The discontinuities are given in terms of $X(t_j)$, t_j and $\nu^{(j)}$; hence the magnitude of the discontinuity in each component of P may be unknown until the problem has been solved. This suggests that a boundary value problem containing $N-1$ intermediate boundaries, where each one is a vector of point-constraints, is equivalent to N two-point boundary value problems in series. Let the entering-corner times be t_j, for $j = 1,2,\ldots, N$. The j-th two-point boundary value problem will extend from t_{j-1} up to t_j. At the beginning of the j-th problem it will be necessary to determine initial values for each P_k $(k = 1,2,\ldots, n)$ which experiences a discontinuity at t_{j-1}. For the components of P which are continuous at t_{j-1} the initial values for the j-th problem are the same as the terminal values for the $j-1$ problem. As the multipliers $\nu^{(j)}$ only appear at the points of discontinuity, they can be dismissed from the discussion. In Chapter 3 it was shown that P_k could be continuous at a point where the trajectory leaves a state-variable constraint boundary. Therefore, each of the N problems has initial and terminal boundary conditions while none have intermediate boundary conditions. Without loss of generality, the following discussion can be restricted to a problem with no intermediate boundary conditions. The procedures for calculating changes in $P(t_0)$ in the "reduced" problem will carry over to the case of N problems in series, containing N sets of initial values $P(t_{j-1}^{+})$, $(j = 1,2,\ldots, N)$.

4.1 The perturbation method for inequality-constrained problems.

For the "reduced" problem of the previous section the following relationships hold, in addition to the condition expressed by Inequality (3.43).

(1) Equations of motion.

$$\dot{X} = H_P^T = F(X,U,t) \tag{4.1}$$

$$-\dot{P} = H_X^T = Q_X^T + F_X^T P + C_X^T M \tag{4.2}$$

where M and U are determined from

$$0 = M_i C_i(X,U,t), \quad (i = 1,2,\ldots, r) \tag{4.3}$$

$$0 = H_U^T = Q_U^T + F_U^T P + C_U^T M \tag{4.4}$$

on $t_0 \le t \le t_N$.

(2) Boundary conditions.

t_0 is known and

$$X(t_0) = X^o \tag{4.5}$$

$$L(X(t_N),t_N) = 0 \tag{4.6}$$

$$P^T(t_N) = R_X(t_N) \tag{4.7}$$

$$R_t(t_N) + H(t_N) + M^T(t_N)\dot{Z}^2(t_N) = 0 \tag{4.8}$$

where

$$R = G(X(t_N),t_N) + \nu^T L(X(t_N),t_N)$$

$$H = Q(X,U,t) + P^T F(X,U,t) + M^T C(X,U,t)$$

$$0 = Z^2 + C(X,U,t)$$

L is an ℓ-dimensional vector. It has been assumed that t_N is unknown.

The terminal conditions, Equation (4.6), represent ℓ algebraic equations in the $n+1$ unknowns $X_1(t_N), X_2(t_N), \ldots, X_n(t_N), t_N$. Assume that these equations are independent in the sense that the $\ell\times(n+1)$ matrix

$$\left[\frac{\partial L}{\partial X}\ \frac{\partial L}{\partial t}\right]$$

evaluated at $t = t_N$, is of rank ℓ. Then ℓ of the variables $X_1, X_2, \ldots, X_n, t_N$ can be found in terms of the remaining $n+1-\ell$ variables, [1]. For notational convenience, assume that $X_1, X_2, \ldots, X_\ell$ are found in terms of $X_{\ell+1}, X_{\ell+2}, \ldots, X_n, t_N$. Thus,

$$X_i = Y_i(X_k, t_N)$$

for

$$i = 1,2,\ldots,\ \ell$$
$$k = \ell+1, \ell+2, \ldots,\ n$$

and therefore the $\ell \times \ell$ matrix

$$\left[\frac{\partial L_i}{\partial X_j}\right] \quad i,\ j = 1,2,\ldots,\ \ell$$

is nonsingular, [1]. It should be noted that if the functions Y_i are readily obtainable then the terminal conditions can be taken as $L_i = X_i - Y_i(X_k, t_N)$ and therefore, the matrix $[\partial L_i/\partial X_j]$ is the $\ell \times \ell$ identity matrix.

Equation (4.7) can be rewritten as

$$P_i = \frac{\partial G}{\partial X_i} + \sum_{j=1}^{\ell} \frac{\partial L_j}{\partial X_i} \nu_j \ ; \quad i = 1,2,\ldots,\ \ell \qquad (4.7'a)$$

$$P_k = \frac{\partial G}{\partial X_k} + \sum_{j=1}^{\ell} \frac{\partial L_j}{\partial X_k} \nu_j \ ; \quad k = \ell+1, \ell+2, \ldots,\ n\ . \quad (4.7'b)$$

The multipliers ν_j can be determined from Equation (4.7'a) since the $\ell \times \ell$ matrix $[\partial L_j/\partial X_i]$ is nonsingular. Thus,

$$\nu_j = \nu_j(X_\alpha, P_\beta, t_N)$$

for

$$\beta, j = 1,2,\ldots, \ell$$

$$\alpha = 1,2,\ldots, n \ .$$

Since ν_j is a function of $X_1, X_2, \ldots, X_n, P_1, P_2, \ldots, P_\ell, t_N$ then Equation (4.7'b) can be expressed as

$$P_k = Y_k(t_N, X_j, P_i)$$

for

$$i = 1,2,\ldots, \ell$$

$$k = \ell+1, \ell+2, \ldots, n$$

$$j = 1,2,\ldots, n \ .$$

Then at $t = t_N$ the following terminal conditions hold:

$$L_i = L_i(X(t_N), t_N) = 0 \qquad (4.6'a)$$

$$J_k = P_k - Y_k(t_N, X_j, P_i) = 0 \qquad (4.6'b)$$

for

$$i = 1,2,\ldots, \ell$$
$$k = \ell+1,\ell+2,\ldots, n$$
$$j = 1,2,\ldots, n .$$

Thus $X_{\ell+1}, X_{\ell+2},\ldots, X_n,P_1,P_2,\ldots, P_\ell$ are unspecified at the terminal time t_N. If $\nu_j = \nu_j(X_\alpha,P_i,t_N)$ is substituted into Equation (4.8), the resulting expression is

$$s = \frac{\partial G}{\partial t} + g(X_\alpha,P_i,t_N) + H + M^T Z^2 = 0 \qquad (4.8')$$

where

$$g(X_\alpha,P_i,t_N) = \sum_{j=1}^{\ell} \frac{\partial L_j}{\partial t}\nu_j(X_\alpha,P_i,t_N)$$

$$\alpha = 1,2,\ldots, n$$
$$i = 1,2,\ldots, \ell .$$

The procedure to be followed in solving the boundary value problem is as follows. Let $P(t_0)$ and t_N be estimated values of the initial Lagrange multipliers and the terminal time, respectively. Equations (4.1) and (4.2) are integrated from t_0 to t_N, with Initial Conditions (4.5) and $P(t_0)$. During the integration process, M and U are calculated from Equations (4.3) and (4.4). At t_N Equations (4.6') and (4.8') will generally not be satisfied. A trajectory (X,P,U,M,t) with these properties will be called a nominal trajectory. Desired changes in Equations (4.6') and (4.8') must be related to changes in t_N and $P(t_0)$ so that a new nominal trajectory can be obtained. This new trajectory is required to satisfy, more closely, the terminal conditions at $t_N + \Delta t_N$ than did the present nominal at t_N. A smaller terminal error norm will indicate this event.

Consider a perturbed trajectory which is "close" to the nominal. The perturbed trajectory $(X+x, P+p, U+u, M+\mu, t)$ will be used to obtain the information needed to generate the new nominal. Since the present nominal trajectory is (X,P,U,M,t), it is necessary only to find (x,p,u,μ,t). Replace X by $X+x$, P by $P+p$, etc. in Equations (4.1) through (4.4). Expand each of the terms in the resulting equations in a Taylor series about the nominal trajectory (X,P,U,M,t), for each value of t, and retain only the first order terms. The resulting linearized equations of perturbed motion are

$$\dot{x} = H_{PX}x + H_{PU}u \tag{4.9}$$

$$\dot{p} = - H_{XX}x - H_{XP}p - H_{XU}u - C_X^T\mu \tag{4.10}$$

$$0 = H_{UU}u + H_{UX}x + H_{UP}p + C_U^T\mu \tag{4.11}$$

$$0 = \mu_i C_i + M_i \left[\frac{\partial C_i}{\partial X} x + \frac{\partial C_i}{\partial U} u \right] \tag{4.12}$$

where

$$i = 1,2,\ldots, r .$$

The solution of the preceding system of equations is called the linearized trajectory. The quantities

$$H_{PX}, C_X, \frac{\partial C_i}{\partial X}, \text{ etc.}$$

are evaluated on the nominal trajectory.

Examination of Equation (4.12) shows that

(1) if $C_i = 0$ $(M_i > 0)$ then

$$\frac{\partial C_i}{\partial X} x + \frac{\partial C_i}{\partial U} u = 0 ,$$

(2) if $C_i < 0$ $(M_i = 0)$ then

$$\mu_i = 0 .$$

This will mean that the linearized trajectory enters and leaves the boundary of a linearized constraint,

$$\frac{\partial C_i}{\partial X} x + \frac{\partial C_i}{\partial U} u = 0 ,$$

at the same times as the nominal trajectory enters and leaves $C_i = 0$. Thus, the perturbed trajectory $(X+x, P+p, U+u, M+\mu, t)$ is forced to travel parallel to (either above or below) a constraint boundary $(C_i = 0)$, between the times that the nominal is on the boundary. Although the perturbed trajectory represents an approximation to a new nominal, this does not imply that successive nominal trajectories necessarily have the same entering- and exiting-corner times.

The quantities u and μ must now be obtained in terms of x and p. Let us suppose that r_1 of the constraints C_i are simultaneously zero at time t. Form the r_1-dimensional vectors E and η: the components of E and η are

$$E_k = \frac{\partial C_{i_k}}{\partial X} x + \frac{\partial C_{i_k}}{\partial U} u$$

$$\eta_k = \mu_{i_k}$$

if $C_{i_k} = 0$ for $k = 1,2,\ldots, r_1$. (If $C_j < 0$ then $\mu_j = 0$.) The terms $C_X^T\mu$ and $C_U^T\mu$, in Equations (4.10) and (4.11), are therefore

$$C_X^T\mu = E_x^T\eta \tag{4.13}$$

$$C_U^T\mu = E_u^T\eta \tag{4.14}$$

and furthermore,

$$E(x,u,t) = 0 \ . \tag{4.15}$$

On substituting Equation (4.14) into Equation (4.11) and recalling that H_{UU} is a positive definite matrix, the relation for u can be determined as

$$u = -H_{UU}^{-1}[H_{UX}x + H_{UP}p + E_u^T\eta] \ . \tag{4.16}$$

Since each component of E is linear in x and in u, Equation (4.15) may be rewritten as

$$E = (E_x)x + (E_u)u = 0 \ . \tag{4.17}$$

Substituting Equation (4.16) into Equation (4.17) and solving for the terms in η leads to

$$(E_u H_{UU}^{-1} E_u^T)\eta = (E_x - E_u H_{UU}^{-1} H_{UX})x - (E_u H_{UU}^{-1} H_{UP})p \ . \tag{4.18}$$

Recall from Chapter 3 that whenever r_1 inequality constraints are simultaneously zero, the $r_1 \times m$ matrix

$$\left[\frac{\partial C_{i_k}}{\partial U_j}\right] \tag{4.19}$$

has rank r_1, where $C_{i_k} = 0$,

$$k = 1,2,\ldots, r_1$$
$$j = 1,2,\ldots, m\ .$$

But the matrix given in (4.19) is E_u. To prove that the $r_1 \times r_1$ matrix multiplying η in Equation (4.18) is nonsingular, let y be an arbitrary, nonzero r_1-dimensional vector. Then

$$y^T(E_u H_{UU}^{-1} E_u^T)y = (y^T E_u)(H_{UU}^{-1})(E_u^T y) = W^T H_{UU}^{-1} W \geq 0 \qquad (4.20)$$

since if H_{UU} is positive definite, so is H_{UU}^{-1}. Since the matrix E_u is of rank r_1 and y is nonzero, $E_u^T y = W \neq 0$, strict inequality holds in (4.20). Therefore, the matrix $E_u H_{UU}^{-1} E_u^T$ is positive definite and can be inverted. Solving for η in Equation (4.18) gives

$$\eta = S^{-1}[Ax + Bp] \qquad (4.21)$$

where

$$S = E_u H_{UU}^{-1} E_u^T \qquad (4.22)$$

$$A = E_x - E_u H_{UU}^{-1} H_{UX} \qquad (4.23)$$

$$B = -E_u H_{UU}^{-1} H_{UP}\ . \qquad (4.24)$$

Equation (4.16) can now be written as

$$u = -H_{UU}^{-1}[(H_{UX} + E_u^T S^{-1} A)x + (H_{UP} + E_u^T S^{-1} B)p] \, . \qquad (4.25)$$

On substituting Equation (4.13) into Equation (4.10) and then substituting Equations (4.21) and (4.25) into Equations (4.9) and (4.10), the resulting equations are

$$\dot{x} = \mathbb{D}_1 x + \mathbb{D}_2 p \qquad (4.26)$$

$$\dot{p} = \mathbb{D}_3 x - \mathbb{D}_1^T p \qquad (4.27)$$

where

$$\mathbb{D}_1 = \Lambda_1 + B^T S^{-1} A \qquad (4.28)$$

$$\mathbb{D}_2 = \Lambda_2 + B^T S^{-1} B \qquad (4.29)$$

$$\mathbb{D}_3 = \Lambda_3 - A^T S^{-1} A \qquad (4.30)$$

and

$$\Lambda_1 = H_{PX} - H_{PU} H_{UU}^{-1} H_{UX} \qquad (4.31)$$

$$\Lambda_2 = -H_{PU} H_{UU}^{-1} H_{UP} \qquad (4.32)$$

$$\Lambda_3 = -H_{XX} + H_{XU} H_{UU}^{-1} H_{UX} \, . \qquad (4.33)$$

Since the matrices $\mathbb{D}_1$, $\mathbb{D}_2$ and $\mathbb{D}_3$ are evaluated along the nominal trajectory, they are functions of t. At time t, if $C_i < 0$ (for $i = 1,2,\ldots, r$) then the terms in A,B, and S do not appear in Equations (4.28), (4.29) and (4.30). Furthermore, Equations (4.31), (4.32) and (4.33) will not contain any terms involving C_X or C_U, because M is zero. In this case the $\mathbb{D}$-matrices reduce to the matrices given by Breakwell et al. [8] for an unconstrained problem. Hence, the $\mathbb{D}$-matrices of Equations (4.26) and (4.27) are generalizations of those obtained by Breakwell. Equations (4.26) and (4.27) are the differential equations which govern the linear perturbations. Boundary conditions for Equations (4.26) and (4.27) must now be found.

Recall that the terminal conditions are generally not satisfied on the nominal trajectory:

$$L_i = L_i(X_j, t_N) \neq 0$$

$$J_k = P_k - Y_k(t_N, X_j, P_i) \neq 0$$

$$s = \frac{\partial G}{\partial t} + g(X_j, P_i, t_N) + H + M^T Z^2 \neq 0$$

for

$$i = 1,2,\ldots, \ell$$

$$k = \ell+1,\ \ell+2,\ldots, n$$

$$j = 1,2,\ldots, n\ .$$

Let

$$L = \begin{bmatrix} L_1 \\ L_2 \\ \cdot \\ \cdot \\ \cdot \\ L_\ell \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} J_{\ell+1} \\ J_{\ell+2} \\ \cdot \\ \cdot \\ \cdot \\ J_n \end{bmatrix} .$$

On replacing X by $X+\Delta X$, P by $P+\Delta P$, t_N by $t_N+\Delta t_N$, etc. in the equations for L, J and s, expanding the resulting expressions in a Taylor series about the terminal values associated with the nominal trajectory and then retaining only the first order terms, the linearized terminal conditions are obtained:

$$\Delta L = (L_X)\Delta X + (L_t)\Delta t_N$$

$$\Delta J = (J_X)\Delta X + (J_P)\Delta P + (J_t)\Delta t_N$$

$$\Delta s = (s_X)\Delta X + (s_P)\Delta P + (s_t)\Delta t_N \; .$$

The terms in parentheses are evaluated at t_N. The subscripts denote partial differentiation with respect to the subscript variable. Some terms in the third equation have been omitted because they are zero on the nominal trajectory. Using the relationships

$$\Delta X = x + \dot{X}\Delta t_N$$

$$\Delta P = p + \dot{P}\Delta t_N$$

in the equations for ΔL, ΔJ and Δs, the following equations can be determined:

$$\Delta L = (L_X)x + (\dot{L})\Delta t_N \tag{4.34}$$

$$\Delta J = (J_X)x + (J_P)p + (\dot{J})\Delta t_N \tag{4.35}$$

$$\Delta s = (s_X)x + (s_P)p + (\dot{s})\Delta t_N \tag{4.36}$$

where

$$\dot{L} = L_X\dot{X} + L_t$$

$$\dot{J} = J_X\dot{X} + J_P\dot{P} + J_t$$

$$\dot{s} = s_X\dot{X} + s_P\dot{P} + s_t \ .$$

Recall that $n-\ell$ of the X_j are free at the terminal time, in addition to ℓ of the P_j. Thus, if $X_i(i=\ell+1,\ldots, n)$ and $P_j(j=1,2,\ldots, \ell)$ are free at the terminal time, the quantities ΔX_i and ΔP_j can form the n-dimensional vector f. Equations (4.34), (4.35) and (4.36) can be solved for x,p and Δt_N in terms of f, ΔL, ΔJ and Δs. Let the solution be expressed as

$$\begin{bmatrix} x(t_N) \\ p(t_N) \\ \Delta t_N \end{bmatrix} = \begin{bmatrix} K_{11}(t_N) & K_{12}(t_N) & K_{13}(t_N) \\ K_{21}(t_N) & K_{22}(t_N) & K_{23}(t_N) \\ K_{31}(t_N) & K_{32}(t_N) & K_{33}(t_N) \end{bmatrix} \begin{bmatrix} f \\ \Delta\Psi \\ \Delta s \end{bmatrix} \tag{4.37}$$

where

K_{11}, K_{12}, K_{21} and K_{22} are $n \times n$ matrices; K_{31}, K_{32}, K_{13}^T and K_{23}^T are $1 \times n$ matrices; K_{33} is a scalar; and

$$\Psi = \begin{bmatrix} L \\ J \end{bmatrix} \quad \text{implying} \quad \Delta\Psi = \begin{bmatrix} \Delta L \\ \Delta J \end{bmatrix} .$$

The equations for $x(t_N)$ and $p(t_N)$ in Equation (4.37) represent the required boundary conditions at t_N for the differential equations given by Equations (4.26) and (4.27). The boundary conditions are given in terms of the desired changes in the terminal conditions. Relating Equations (4.37) through Equations (4.26) and (4.27) to changes in $P(t_0)$ and changes in t_N must now be accomplished.

Equations (4.26) and (4.27) may be written as

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \\ \mathbb{D}_3 & -\mathbb{D}_1^T \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} . \tag{4.38}$$

Equation (4.37) is equivalent to

$$\begin{bmatrix} x(t_N) \\ \\ p(t_N) \end{bmatrix} = \begin{bmatrix} K_{11}(t_N) & K_{12}(t_N) & K_{13}(t_N) \\ \\ K_{21}(t_N) & K_{22}(t_N) & K_{23}(t_N) \end{bmatrix}\begin{bmatrix} f \\ \Delta\Psi \\ \Delta s \end{bmatrix} \tag{4.39}$$

and

$$\Delta t_N = K_{31}(t_N)f + K_{32}(t_N)\Delta\Psi + K_{33}(t_N)\Delta s. \tag{4.40}$$

Two methods for the solution of Equation (4.38) will be given. Both methods utilize the properties of a system of linear first order ordinary differential equations. The essential feature of each method is the generation of a matrix whose columns are solutions of Equation (4.38). Either method can form part of a computer algorithm. Each algorithm contains (1) the integration of Equations (4.1) and (4.2) from t_0 to t_N, with U and M calculated from Equations (4.3) and (4.4), and (2) the application of one of the following methods.

4.2 Method 1.

Let the $2n \times 2n$ matrix $\Theta(t)$ be a fundamental matrix for Equation (4.38), [11]. On expressing $\Theta(t)$ in block form, $x(t)$ and $p(t)$ can be obtained:

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Theta_{11}(t) & \Theta_{12}(t) \\ \Theta_{21}(t) & \Theta_{22}(t) \end{bmatrix} \begin{bmatrix} x(t_N) \\ p(t_N) \end{bmatrix} \tag{4.41}$$

where

$$\Theta_{11}(t_N) = \Theta_{22}(t_N) = I \text{ , the } n \times n \text{ identity matrix,}$$
$$\Theta_{12}(t_N) = \Theta_{21}(t_N) = 0 \text{ , the } n \times n \text{ null matrix.}$$

On substituting Equation (4.39) into Equation (4.41) and setting $t = t_0$, the linear perturbations at t_0 are determined. Thus,

$$\begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} = \begin{bmatrix} K_{11}(t_0) & K_{12}(t_0) & K_{13}(t_0) \\ K_{21}(t_0) & K_{22}(t_0) & K_{23}(t_0) \end{bmatrix} \begin{bmatrix} f \\ \Delta\Psi \\ \Delta s \end{bmatrix} \tag{4.42}$$

where

$$K_{1j}(t_0) = \Theta_{11}(t_0)K_{1j}(t_N) + \Theta_{12}(t_0)K_{2j}(t_N)$$

$$K_{2j}(t_0) = \Theta_{21}(t_0)K_{1j}(t_N) + \Theta_{22}(t_0)K_{2j}(t_N)$$

$j = 1,2,3$.

Since $X(t_0)$ is known, $x(t_0) = 0$, and Equation (4.42) reduces to

$$0 = K_{11}(t_0)f + K_{12}(t_0)\Delta\Psi + K_{13}(t_0)\Delta s \tag{4.43}$$

$$p(t_0) = K_{21}(t_0)f + K_{22}(t_0)\Delta\Psi + K_{23}(t_0)\Delta s \tag{4.44}$$

The unknowns in Equations (4.43) and (4.44) are f and $p(t_0)$. If $K_{11}(t_0)$ is nonsingular, f can be determined from Equation (4.43), $p(t_0)$ found from Equation (4.44) and Δt_N calculated from Equation (4.40). (The matrix $K_{11}(t_0)$ is related to the conditions for a conjugate point. The test for conjugate points is given in Section 4.4.) Supposing that $K_{11}(t_0)$ is nonsingular, then

$$p(t_0) = W_{11}\Delta\Psi + W_{12}\Delta s$$

$$\Delta t_N = W_{21}\Delta\Psi + W_{22}\Delta s$$

where

$$W_{11} = K_{22}(t_0) - K_{21}(t_0)[K_{11}(t_0)]^{-1}K_{12}(t_0)$$

$$W_{12} = K_{23}(t_0) - K_{21}(t_0)[K_{11}(t_0)]^{-1}K_{13}(t_0)$$

$$W_{21} = K_{32}(t_N) - K_{31}(t_N)[K_{11}(t_0)]^{-1}K_{12}(t_0)$$

$$W_{22} = K_{33}(t_N) - K_{31}(t_N)[K_{11}(t_0)]^{-1}K_{13}(t_0) \ .$$

The values of $\Delta\Psi$ and Δs are determined by

$$\Delta\Psi = -\alpha\Psi(t_N) \ , \quad \text{and}$$

$$\Delta s = -\beta s(t_N) \ ,$$

where Ψ and s are evaluated on the nominal trajectory. The factors α and β are scaling constants, $0<\alpha, \beta \leq 1$. They are chosen so that $\|p(t_0)\|$ and $|\Delta t_N|$ will not be too large. The magnitudes of $p(t_0)$ and t_N are required to be small in order that the perturbed trajectory will remain "close" to the nominal trajectory. The appropriate choice for the values of α and β must be determined empirically.

The new values of $P(t_0)$ and t_N are formed as follows: $P(t_0)+p(t_0)$ replaces $P(t_0)$, and $t_N+\Delta t_N$ replaces t_N. With these new values, the iterative cycle is repeated by finding a new nominal trajectory. The new nominal should yield a smaller terminal-constraint error than the previous nominal. The iterative process may be stopped whenever the terminal errors, $\| \Psi \|$ and $|s|$, are small enough so that the current nominal trajectory may be accepted as the optimal trajectory.

4.3 Method 2.

Let the matrix

$$\begin{bmatrix} \Phi_3(t) & \Phi_1(t) \\ \Phi_4(t) & \Phi_2(t) \end{bmatrix}$$

be a fundamental matrix for Equation (4.38). A solution of Equation (4.38) is given by

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Phi_3(t) & \Phi_1(t) \\ \Phi_4(t) & \Phi_2(t) \end{bmatrix} \begin{bmatrix} x(t_0) \\ p(t_0) \end{bmatrix} \qquad (4.45)$$

with

$$\Phi_3(t_0) = \Phi_2(t_0) = I \text{ , the } n\times n \text{ identity matrix}$$
$$\Phi_1(t_0) = \Phi_4(t_0) = 0 \text{ , the } n\times n \text{ null matrix .}$$

Since $x(t_0) = 0$, Equation (4.45) reduces to

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Phi_1(t)p(t_0) \\ \Phi_2(t)p(t_0) \end{bmatrix} ,$$

and therefore part of the fundamental matrix, namely

$$\begin{bmatrix} \Phi_3(t) \\ \Phi_4(t) \end{bmatrix} ,$$ need not be determined.

At $t = t_N$,

$$\begin{bmatrix} x(t_N) \\ p(t_N) \end{bmatrix} = \begin{bmatrix} \Phi_1(t_N)p(t_0) \\ \Phi_2(t_N)p(t_0) \end{bmatrix} . \qquad (4.46)$$

But Equation (4.39) gives $x(t_N)$ and $p(t_N)$ in terms of f, $\Delta\Psi$ and Δs. Equating Equation (4.46) to Equation (4.39) and rearranging terms so that the unknowns, f and $p(t_0)$, are on the same side of the new equation, gives

$$\begin{bmatrix} K_{11}(t_N) & -\Phi_1(t_N) \\ K_{21}(t_N) & -\Phi_2(t_N) \end{bmatrix} \begin{bmatrix} f \\ p(t_0) \end{bmatrix} = \begin{bmatrix} -K_{12}(t_N)\Delta\Psi - K_{13}(t_N)\Delta s \\ -K_{22}(t_N)\Delta\Psi - K_{23}(t_N)\Delta s \end{bmatrix} \qquad (4.47)$$

where $\Delta\Psi$ and Δs are $-\alpha\Psi(t_N)$ and $-\beta s(t_N)$ respectively. If the matrix on the left hand side of Equation (4.47) is nonsingular, f and $p(t_0)$ can be found; from Equation (4.40) Δt_N can be determined; then the new values of $P(t_0)$ and t_N can be formed in the same way as in Method 1. If the matrix is singular, the generalized matrix inverse [12] can be used to solve Equation (4.47). This is done in the following manner. Let the linear system given in Equation (4.47) be represented by $\Omega y = b$. Since the matrix Ω is singular, there is not a unique solution y. The generalized inverse of Ω, Ω^+, is found, [12]. The solution accepted is $\bar{y} = \Omega^+ b$. This solution represents the best approximation to y, in the least squares sense. In either case (Ω singular or nonsingular), the stopping criterion for the iterative process is exactly the same as that of Method 1.

One practical advantage of Method 2 over Method 1 is the smaller number of different initial conditions which must be integrated to form the matrix used in finding f and $p(t_0)$. Method 1 requires the integration of $2n$ initial conditions. With Method 2 requiring only n initial conditions, n initial conditions for each iteration cycle need not be integrated. The amount of computer time is reduced by half. A disadvantage of Method 2 is the possible singularity of the matrix in Equation (4.47). Use of the generalized matrix inverse may appreciably lower the rate of convergence so that the total amount of computer time becomes greater than that required by Method 1. To determine the more efficient Method one can obtain the convergence rates by several test computer runs.

This procedure, Method 2, is similar to the one given by Breakwell *et al*. [8] for unconstrained problems.

4.4 Testing for conjugate points.

Recall, from Section 3.6, the conjecture on the procedure of Breakwell and Ho [7] for determining the existence of points conjugate to the terminal time. It was conjectured that their procedure was applicable to problems with inequality constraints, provided modifications were made to account for the effect of the constraints. The matrix which is to be tested is

$$K_{11}(t) = \Theta_{11}(t)K_{11}(t_N) + \Theta_{12}(t)K_{21}(t_N)$$

where $K_{11}(t_N)$ and $K_{21}(t_N)$ are defined by Equation (4.37); $\Theta_{11}(t)$ and $\Theta_{12}(t)$ are defined by Equation (4.41). Now, in an inequality-constrained problem the effects of the constraints influence $\Theta_{11}(t)$ and $\Theta_{12}(t)$ because these two matrices are part of a fundamental matrix for Equation (4.38). If the constraints were absent then $K_{11}(t)$ reduces to the matrix considered by Breakwell and Ho. Thus, $K_{11}(t)$ is the matrix to be examined for the case of inequality-constrained problems (see Section 3.6).

A different procedure is used to determine the existence of points conjugate to the initial time, t_0. Consider ΔV for a trajectory in which $\delta' V = 0$, H_{UU} is positive definite, $p = \dot{p} = 0$ and $\mu = 0$. The terminal time is considered to be fixed. Then

$$\Delta V = \frac{1}{2}\delta'' V = \frac{1}{2}[x^T R_{XX} x]_{t_N} + \int_{t_0}^{t_N} \omega(x,u,t)dt \tag{4.48}$$

where

$$\omega(x,u,t) = \frac{1}{2}x^T H_{XX} x + x^T H_{XU} u + \frac{1}{2}u^T H_{UU} u \ . \tag{4.49}$$

The following conditions are imposed on x and u:

$$\dot{x} = H_{PX} x + H_{PU} u$$

$$x(t_0) = 0$$

$$x(t_N) = b \ , \quad b \text{ is unspecified}$$

$$c_i \le 0 \quad i = 1,2,\ldots, r$$

where $\quad c_i < 0 \ \text{ if } \ C_i < 0$

$$c_i = \left(\frac{\partial C_i}{\partial X}\right)x + \left(\frac{\partial C_i}{\partial U}\right)u = 0 \quad \text{if} \quad C_i = 0$$

or $z_i^2 + c_i = 0$. Form the functional χ by adjoining the above relationships to ΔV; then find $\delta'\chi$ by the same procedure employed in Chapter 3. (χ is an extremum when $\delta'\chi = 0$.)

$$\begin{aligned}\delta'\chi = {} & [(x^T R_{XX} - \lambda^T + \kappa^T)(\delta x) + (x-b)^T(\delta\kappa)]_{t_N} \\ & + \int_{t_0}^{t_N} \{(\omega_x + \lambda^T H_{PX} + \rho^T c_x + \dot{\lambda}^T)(\delta x) \\ & \quad + (\omega_u + \lambda^T H_{PU} + \rho^T c_u)(\delta u) \\ & \quad + (H_{PX}x + H_{PU}u - \dot{x})^T(\delta\lambda) \\ & \quad + (c + z^2)^T(\delta\rho) + 2\sum_{i=1}^{r} \rho_i z_i(\delta z_i)\}dt \ .\end{aligned}$$

On setting $\delta'\chi = 0$, the following conditions are obtained:

$$[(x^T R_{XX} - \lambda^T + \kappa^T)(\delta x)]_{t_N} = 0$$

$$x(t_N) = b \ .$$

$$\dot{x} = H_{PX}x + H_{PU}u \tag{4.50}$$

$$\dot{\lambda} = -H_{XX}x - H_{XU}u - H_{XP}\lambda - c_x^T\rho \tag{4.51}$$

$$0 = H_{UU}u + H_{UX}x + H_{UP}\lambda + c_u^T\rho \tag{4.52}$$

$$0 = \rho_i z_i \ , \quad (i = 1,2,\dots, r) \tag{4.53}$$

By an argument similar to that used in Section 4.1, the following differential equation is obtained:

$$\frac{d}{dt}\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \\ \mathbb{D}_3 & -\mathbb{D}_1^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}$$

where the $\mathbb{D}$-matrices are defined in Equations (4.28), (4.29) and (4.30). Furthermore u is given by an equation of the same form as Equation (4.25). Using this equation, replace u in the expression for $\omega(x,u,t)$ to give

$$\omega = -\frac{1}{2}\, x^T\mathbb{D}_3 x - \frac{1}{2}\lambda^T\mathbb{D}_2\lambda \ .$$

Now

$$\frac{d}{dt}(\lambda^T x) = \dot{\lambda}^T x + \lambda^T\dot{x} = (x^T\mathbb{D}_3^T - \lambda^T\mathbb{D}_1)x + \lambda^T(\mathbb{D}_1 x + \mathbb{D}_2\lambda)$$

$$= x^T\mathbb{D}_3 x + \lambda^T\mathbb{D}_2\lambda$$

since $\mathbb{D}_3$ is a symmetric matrix. Thus,

$$\omega = -\frac{1}{2}\frac{d}{dt}(\lambda^T x)$$

and therefore

$$\Delta V = [\tfrac{1}{2}x^T R_{XX} x]_{t_N} - \frac{1}{2}\int_{t_0}^{t_N} \frac{d}{dt}(\lambda^T x)\,dt$$

$$= \tfrac{1}{2}[x^T R_{XX} x - \lambda^T x]_{t_N}$$

since $x(t_0) = 0$. Define the matrices $\Phi_1(t)$ and $\Phi_2(t)$ as in Section 4.3. Since $x(t_0) = 0$,

$$x(t) = \Phi_1(t)\lambda_0$$

$$\lambda(t) = \Phi_2(t)\lambda_0$$

with $\lambda_0 = \lambda(t_0)$

$$\Phi_1(t_0) = 0 \;, \quad n\times n \text{ null matrix}$$

$$\Phi_2(t_0) = I \;, \quad n\times n \text{ identity matrix} .$$

Then ΔV can be written as

$$\Delta V = \frac{1}{2} \lambda_0^T [\Phi_1^T(t_N) R_{XX} \Phi_1(t_N) - \Phi_2^T(t_N)\Phi_1(t_N)]\lambda_0 .$$

Define the matrix Π as

$$\Pi = \Phi_1^T(t_N) R_{XX} \Phi_1(t_N) - \Phi_2^T(t_N)\Phi_1(t_N) .$$

Thus

$$\Delta V = \frac{1}{2} \lambda_0^T \Pi \lambda_0 .$$

Now the term $\Phi_1^T(t_N) R_{XX} \Phi_1(t_N)$ is symmetric. The matrices $\Phi_1(t)$ and $\Phi_2(t)$ satisfy the differential equations

$$\dot{\Phi}_1 = \mathbb{D}_1 \Phi_1 + \mathbb{D}_2 \Phi_2$$

$$\dot{\Phi}_2 = \mathbb{D}_3 \Phi_1 - \mathbb{D}_1^T \Phi_2$$

and therefore the matrix $\Phi_2^T \Phi_1$ satisfies the differential equation

$$\frac{d}{dt}(\Phi_2^T \Phi_1) = \Phi_1^T \mathbb{D}_3 \Phi_1 + \Phi_2^T \mathbb{D}_2 \Phi_2 = \frac{d}{dt}(\Phi_1^T \Phi_2) .$$

Therefore $\Phi_2^T \Phi_1$ is symmetric. Thus, the matrix Π has real eigenvalues. Finally, $\Delta V > 0$ for arbitrary $\lambda_0 \neq 0$ if the real, symmetric matrix Π is positive definite. If Π has negative eigenvalues then there exists a λ_0 such that $\Delta V < 0$

and thus there is a conjugate point on the trajectory. The test for the existence of points conjugate to the initial time t_0 is reduced to determining if the real, symmetric matrix Π is positive definite.

It should be noted that any segment of an optimum trajectory must also be optimum. Hence, the above procedure must also apply to each t_1, $t_0 \leq t_1 < t_N$, with t_1 replacing t_0 as the initial time.

In this chapter computational algorithms, based on a perturbation method, were devised for inequality-constrained optimization problems. Testing the algorithms is accomplished by the numerical solution of a constrained nonlinear problem. The chosen problem and the numerical results are presented in Chapter 5.

CHAPTER 5

NUMERICAL SOLUTION OF A CONSTRAINED NONLINEAR PROBLEM

5.1 The constrained nonlinear problem.

A nonlinear problem originally studied by Fowler [20] was selected to test the algorithms of Chapter 4. The problem, a minimum time low-thrust Earth-Mars transfer at constant mass-flow rate, was modified by introducing inequality constraints. A brief discussion of the original problem appears in Appendix A.

Considered as a nonlinear optimization problem with inequality constraints, the modified problem can be stated as follows. Select $U_3(t)$ and $U_4(t)$, $0 \leq t \leq t_N$, so as to minimize the functional

$$\Gamma = U_1 t_N \tag{5.1}$$

subject to the differential equations

$$\dot{X}_1 = -\frac{\gamma X_4}{\rho^3} + \frac{U_1 U_2}{1-U_1 t}[\cos U_3 \cos U_4] \tag{5.2}$$

$$\dot{X}_2 = -\frac{\gamma X_5}{\rho^3} + \frac{U_1 U_2}{1-U_1 t}[\cos U_3 \sin U_4] \tag{5.3}$$

$$\dot{X}_3 = -\frac{\gamma X_6}{\rho^3} + \frac{U_1 U_2}{1-U_1 t}[\sin U_3] \tag{5.4}$$

$$\dot{X}_4 = X_1 \tag{5.5}$$

$$\dot{X}_5 = X_2 \tag{5.6}$$

$$\dot{X}_6 = X_3 \tag{5.7}$$

where $\rho^2 = X_4^2 + X_5^2 + X_6^2$, U_1 and U_2 are constants (they are β and $|c|$ in Appendix A); the terminal conditions

$$L_i = X_i - Y_i(t_N) = 0 \tag{5.8}$$

for $i = 1,2,3,4,5,6$ (where $Y_i(t_N)$ can be determined from Equation (A.13) in Appendix A); the inequality constraints

$$C_1 = U_3^2 - a_3^2 \leq 0 \tag{5.9}$$

$$C_2 = (U_4 - a_1)^2 - a_2^2 \leq 0 \tag{5.10}$$

where a_1, a_2 and a_3 are constants; and the initial conditions are given by Equations (A.14) through (A.19) in Appendix A, for $X_1(0)$ to $X_6(0)$ respectively. For this problem, $m = 2$, $n = 6$, $\ell = 6$ and $\Psi = L$. The variational Hamiltonian for the inequality-constrained problem is

$$H = P_1\left[-\frac{\gamma X_4}{\rho^3} + \frac{U_1 U_2}{1-U_1 t}\cos U_3 \cos U_4\right]$$

$$+ P_2\left[-\frac{\gamma X_5}{\rho^3} + \frac{U_1 U_2}{1-U_1 t}\cos U_3 \sin U_4\right]$$

$$+ P_3\left[-\frac{\gamma X_6}{\rho^3} + \frac{U_1 U_2}{1-U_1 t}\sin U_3\right]$$

$$+ P_4 X_1 + P_5 X_2 + P_6 X_3$$

$$+ M_1[U_3^2 - a_3^2] + M_2[(U_4 - a_1)^2 - a_2^2] \,. \tag{5.11}$$

As neither C_1 nor C_2 explicitly contains X, the Lagrange multipliers, P_k, will satisfy the differential equations

$$\dot{P}_1 = -P_4 \tag{5.12}$$

$$\dot{P}_2 = -P_5 \tag{5.13}$$

$$\dot{P}_3 = -P_6 \tag{5.14}$$

$$\dot{P}_4 = \frac{\gamma}{\rho^3} P_1 + bX_4 \tag{5.15}$$

$$\dot{P}_5 = \frac{\gamma}{\rho^3} P_2 + bX_5 \tag{5.16}$$

$$\dot{P}_6 = \frac{\gamma}{\rho^3} P_3 + bX_6 \tag{5.17}$$

where

$$b = -\frac{3\gamma}{\rho^5}[P_1X_4 + P_2X_5 + P_3X_6] \tag{5.18}$$

The boundary conditions that P must satisfy are

$$P(t_N) = \nu \tag{5.19}$$

$$U_1 + \nu^T L_t(t_N) + H(t_N) = 0 \tag{5.20}$$

where ν is unknown. The optimal unconstrained controls are given by

$$\sin U_3 = \frac{-P_3}{\sqrt{P_1^2 + P_2^2 + P_3^2}} \tag{5.21}$$

$$\cos U_3 = \sqrt{\frac{P_1^2 + P_2^2}{P_1^2 + P_2^2 + P_3^2}} \tag{5.22}$$

$$\sin U_4 = \frac{-P_2}{\sqrt{P_1^2 + P_2^2}} \tag{5.23}$$

$$\cos U_4 = \frac{-P_1}{\sqrt{P_1^2 + P_2^2}} \tag{5.24}$$

The multipliers M_1 and M_2 are calculated from

$$M_1 = \begin{cases} \frac{1}{2}\left(\frac{U_1U_2}{1-U_1t}\right) \frac{1}{U_3} [(P_1\cos U_4 + P_2\sin U_4)\sin U_3 - P_3\cos U_3]; & \text{if } C_1 = 0 \\ 0 ; & \text{if } C_1 < 0 \end{cases} \quad (5.25)$$

$$M_2 = \begin{cases} \frac{1}{2}\left(\frac{U_1U_2}{1-U_1t}\right)\left(\frac{\cos U_3}{U_4-a_1}\right) [P_1\sin U_4 - P_2\cos U_4]; & \text{if } C_2 = 0 \\ 0 ; & \text{if } C_2 < 0 \end{cases} \quad (5.26)$$

By Inequality (3.46), $M_1 \geq 0$ and $M_2 \geq 0$. The second partial derivatives of H are given below.

$$H_{XX} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{H}_{XX} \end{bmatrix} \quad (5.27)$$

Each block in Equation (5.27) is a 3×3 matrix.

$$\bar{H}_{XX} = \begin{bmatrix} \frac{\partial^2 H}{\partial X_4^2} & \frac{\partial^2 H}{\partial X_4 \partial X_5} & \frac{\partial^2 H}{\partial X_4 \partial X_6} \\ & \frac{\partial^2 H}{\partial X_5^2} & \frac{\partial^2 H}{\partial X_5 \partial X_6} \\ \text{symmetric} & & \frac{\partial^2 H}{\partial X_6^2} \end{bmatrix}$$

The elements of $\bar{H}_{XX}$ are

$$\frac{\partial^2 H}{\partial X_j^2} = \frac{3\gamma}{\rho^5} X_j P_{j-3} - X_j \frac{\partial b}{\partial X_j} - b \qquad (5.28)$$

$$(j = 4,5,6)$$

$$\frac{\partial^2 H}{\partial X_k \partial X_j} = \frac{3\gamma}{\rho^5} X_k P_{j-3} - X_j \frac{\partial b}{\partial X_k} \qquad (5.29)$$

$$(k = 4,\ j = 5,6;\ k = 5,\ j = 6)$$

where b is given by Equation (5.18).

$$H_{XP} = \begin{bmatrix} 0 & I \\ \bar{H}_{XP} & 0 \end{bmatrix} \qquad (5.30)$$

Each block in Equation (5.30) is a 3×3 matrix.

$$\bar{H}_{XP} = \begin{bmatrix} \frac{\partial^2 H}{\partial X_4 \partial P_1} & \frac{\partial^2 H}{\partial X_4 \partial P_2} & \frac{\partial^2 H}{\partial X_4 \partial P_3} \\ & \frac{\partial^2 H}{\partial X_5 \partial P_2} & \frac{\partial^2 H}{\partial X_5 \partial P_3} \\ \text{symmetric} & & \frac{\partial^2 H}{\partial X_6 \partial P_3} \end{bmatrix}$$

The elements of $\bar{H}_{XP}$ are given by

$$\frac{\partial^2 H}{\partial X_j \partial P_{j-3}} = -\frac{\gamma}{\rho^3} + \frac{3\gamma}{\rho^5} X_j^2 \ , \ (j = 4,5,6) \tag{5.31}$$

$$\frac{\partial^2 H}{\partial X_k \partial P_j} = \frac{3\gamma}{\rho^5} X_k \ X_{j+3} \qquad \begin{matrix} (k = 4, \ j = 5,6) \\ (k = 5, \ j = 6) \ . \end{matrix} \tag{5.32}$$

The 6×2 matrix H_{XU} is identically zero.

$$H_{XU} \equiv 0 \tag{5.33}$$

The 2×2 matrix H_{UU} is given by

$$H_{UU} = \begin{bmatrix} \dfrac{\partial^2 H}{\partial U_3^2} & \dfrac{\partial^2 H}{\partial U_3 \partial U_4} \\ & \\ \dfrac{\partial^2 H}{\partial U_3 \partial U_4} & \dfrac{\partial^2 H}{\partial U_4^2} \end{bmatrix} \tag{5.34}$$

where

$$\frac{\partial^2 H}{\partial U_3^2} = - \left(\frac{U_1 U_2}{1-U_1 t}\right)[(P_1 \cos U_4 + P_2 \sin U_4)\cos U_3 + P_3 \sin U_3] + 2M_1 \tag{5.35}$$

$$\frac{\partial^2 H}{\partial U_3 \partial U_4} = \left(\frac{U_1 U_2}{1-U_1 t}\right)[P_1 \sin U_4 - P_2 \cos U_4]\sin U_3 \tag{5.36}$$

$$\frac{\partial^2 H}{\partial U_4^2} = -\left(\frac{U_1 U_2}{1-U_1 t}\right)[P_1 \cos U_4 + P_2 \sin U_4]\cos U_3 + 2M_2 \tag{5.37}$$

The 6×2 matrix H_{PU} is given by

$$H_{PU} = \begin{bmatrix} \bar{H}_{PU} \\ 0 \end{bmatrix} \tag{5.38}$$

where each block is a 3×2 matrix, and

$$\bar{H}_{PU} = \begin{bmatrix} \dfrac{\partial^2 H}{\partial P_1 \partial U_3} & \dfrac{\partial^2 H}{\partial P_1 \partial U_4} \\[2ex] \dfrac{\partial^2 H}{\partial P_2 \partial U_3} & \dfrac{\partial^2 H}{\partial P_2 \partial U_4} \\[2ex] \dfrac{\partial^2 H}{\partial P_3 \partial U_3} & \dfrac{\partial^2 H}{\partial P_3 \partial U_4} \end{bmatrix}$$

with

$$\frac{\partial^2 H}{\partial P_1 \partial U_3} = -\left(\frac{U_1 U_2}{1-U_1 t}\right) \sin U_3 \cos U_4 \tag{5.39}$$

$$\frac{\partial^2 H}{\partial P_2 \partial U_3} = -\left(\frac{U_1 U_2}{1-U_1 t}\right) \sin U_3 \sin U_4 \tag{5.40}$$

$$\frac{\partial^2 H}{\partial P_3 \partial U_3} = \left(\frac{U_1 U_2}{1-U_1 t}\right) \cos U_3 \tag{5.41}$$

$$\frac{\partial^2 H}{\partial P_1 \partial U_4} = -\left(\frac{U_1 U_2}{1-U_1 t}\right) \cos U_3 \sin U_4 \tag{5.42}$$

$$\frac{\partial^2 H}{\partial P_2 \partial U_4} = \left(\frac{U_1 U_2}{1-U_1 t}\right)\cos U_3 \cos U_4 \tag{5.43}$$

$$\frac{\partial^2 H}{\partial P_3 \partial U_4} = 0 \ . \tag{5.44}$$

The $\mathbb{D}$-matrices, Equations (4.28), (4.29) and (4.30), are given at time t for the four possible cases, as follows.

(1) $C_1 < 0$ and $C_2 < 0$. With $M_1 = M_2 = 0$, it follows that

$$\left.\begin{aligned} \mathbb{D}_1 &= H_{PX} \\ \mathbb{D}_2 &= -H_{PU} W_0 H_{UP} \\ \mathbb{D}_3 &= -H_{XX} \end{aligned}\right\} \tag{5.45}$$

where $W_0 = H_{UU}^{-1}$.

(2) $C_1 = 0$ and $C_2 < 0$. Here $M_2 = 0$ but $M_1 \geq 0$. Let $E = (2U_3)u_3$, where U_3 is the value obtained from $C_1 = 0$; then

$$E_u = [2U_3, \ 0]$$
$$E_x = 0 \ .$$

Equations (4.21), (4.22) and (4.23) then can be determined as ($E_x = 0$ and $H_{UX} = 0$)

$$S = 4U_3^2 h_{11}$$

$$A = 0$$

$$B = -[2U_3 \;,\; 0]H_{UU}^{-1}H_{UP}$$

where $$H_{UU}^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} .$$

Then the $\mathbb{D}$-matrices are

$$\left.\begin{aligned} \mathbb{D}_1 &= H_{PX} \\ \mathbb{D}_2 &= -H_{PU}W_1H_{UP} \\ \mathbb{D}_3 &= -H_{XX} \end{aligned}\right\} \tag{5.46}$$

where the 2×2 matrix W_1 is

$$W_1 = \begin{bmatrix} 0 & 0 \\ 0 & \left(\dfrac{\partial^2 H}{\partial U_4^2}\right)^{-1} \end{bmatrix} .$$

(3) $C_1<0$ and $C_2=0$. For this case $M_1=0$ and $M_2\geq0$. The $\mathbb{D}$-matrices are

$$\left.\begin{aligned} \mathbb{D}_1 &= H_{PX} \\ \mathbb{D}_2 &= -H_{PU} W_2 H_{UP} \\ \mathbb{D}_3 &= -H_{XX} \end{aligned}\right\} \qquad (5.47)$$

where the 2×2 matrix W_2 is

$$W_2 = \begin{bmatrix} \left(\dfrac{\partial^2 H}{\partial U_3^2}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

(4) $C_1 = 0$ and $C_2 = 0$.

Both multipliers, M_1 and M_2, may be non-zero. The matrices are

$$\left.\begin{aligned} \mathbb{D}_1 &= H_{PX} \\ \mathbb{D}_2 &= -H_{PU} W_3 H_{UP} \\ \mathbb{D}_3 &= -H_{XX} \quad . \end{aligned}\right\} \qquad (5.48)$$

Where $W_3 = 0$.

For each of the four cases, only $\mathbb{D}_2$ is changed. In fact the expression for $\mathbb{D}_2$ can be written as

$$\mathbb{D}_2 = -H_{PU} W_i H_{UP}$$

where W_i may be H_{UU}^{-1} , W_1 , W_2 , or W_3 for cases (1), (2), (3) or (4) respectively.

The linearized boundary conditions at t_N are

$$\Delta\Psi = x + (\dot{X} - \dot{Y})\Delta t_N$$

$$\Delta s = (H_X)x + (H_P)p - (\dot{Y}^T)\Delta P$$

$$+ (H_t - P^T\ddot{Y} + H_X\dot{X} + H_P\dot{P})\Delta t_N$$

$$p = \Delta P - (\dot{P})\Delta t_N$$

where the quantities in parentheses are evaluated at t_N.

Set f equal to ΔP and solve for x, p and Δt_N terms of f, $\Delta\Psi$ and Δs. The resulting solution is in the form of Equation (4.37) with

$$K_{11}(t_N) = K_{33}(t_N)[(\dot{X}-\dot{Y})(\dot{X}-\dot{Y})^T] \tag{5.49}$$

$$K_{12}(t_N) = I + K_{33}(t_N)[(\dot{X}-\dot{Y})(H_X)] \tag{5.50}$$

$$K_{13}(t_N) = -K_{33}(t_N)[(\dot{X}-\dot{Y})] \tag{5.51}$$

$$K_{21}(t_N) = I + K_{33}(t_N)[(\dot{P})(\dot{X}-\dot{Y})^T] \tag{5.52}$$

$$K_{22}(t_N) = K_{33}(t_N)[(\dot{P})(H_X)] \tag{5.53}$$

$$K_{23}(t_N) = -K_{33}(t_N)[(\dot{P})] \tag{5.54}$$

$$K_{31}(t_N) = -K_{33}(t_N)[(\dot{X}-\dot{Y})^T] \tag{5.55}$$

$$K_{32}(t_N) = -K_{33}(t_N)[(H_X)] \tag{5.56}$$

$$K_{33}(t_N) = (H_t-P^T\ddot{Y}+H_X\dot{Y})^{-1} \quad . \tag{5.57}$$

I is the 6×6 identity matrix.

5.2 Numerical experiments.

For numerical solution, two computer programs were written in FORTRAN-63 for the Control Data Corporation 1604 Computer at The University of Texas. Method 1 was used in one program while Method 2 was used in the second. In both programs the numerical integrations were carried out using an Adams Predictor-Corrector Procedure (predictor truncation error $O(h^5)$; corrector truncation error $O(h^6)$; where h is the step-size) with a Runge-Kutta starter (truncation error $O(h^5)$) and with partial double-precision arithmetic. Previous experience with the systems of differential equations for this Earth-Mars transfer problem has shown that a step-size of approximately 1 day was sufficient to control the growth of round-off errors and truncation errors. The terminal time was approximately 176 days and therefore the step-size was determined by $h = t_N/176$. Matrix inversions and solutions to linear algebraic systems were implemented by a Gaussian Elimination method, with row pivioting, in double-precision arithmetic. The first part of a study of low-thrust guidance methods at The University of Texas [45] involved the generation of an unconstrained trajectory for the problem in Section 5.1. $P(t_0)$ and t_N for this trajectory were used as initial approximations in order to check-out the computer programs.

Several constrained trajectories with different values for the constants a_1, a_2 and a_3 in Inequalities (5.9) and (5.10) were calculated using both Method 1 and Method 2. Convergence to the same terminal error norm was achieved with the same number of iterations for both methods. Of the two, Method 2 is preferred because it required only half the computer time needed by Method 1.

For each trajectory the scaling factors α and β in $\Delta\Psi = -\alpha\Psi(t_N)$ and $\Delta s = -\beta s(t_N)$ were fixed at $\alpha=\beta=1$. With these values of α and β, it was found that the methods diverged whenever the constraint levels were lowered by too great an amount. Consequently the following procedure was adopted. The constants a_1 and a_2 in Inequality (5.10) were only changed slightly between the different trajectories so that the solution for one constrained problem served as a good initial guess for the next problem. In this manner a series of different constrained trajectories were quickly generated, (see Table 1).

Some experimental results for five different constrained trajectories are listed in Table 1. The bounds on the control variables signify the minimum and maximum values which U_3 and U_4 could attain, when restricted by the constraints given by Inequalities (5.9) and (5.10). Table 2 gives the terminal time, t_N, and the norms of the terminal errors, $\|\Psi\|$ and $|s|$, for each iteration needed to obtain Trajectory 5. Comparison of the initial and final values of the terminal error norms and the number of iterations required, shows that the algorithms of Chapter 4 provide a rapidly converging method for the solution of constrained optimization problems.

TABLE 1

Trajectory	Bounds on U_3		Bounds on U_4		$\|\Psi\|$		t_N		Number of Iterations
	Lower	Upper	Lower	Upper	Initial	Final	Initial	Final	
1	-1000	1000	0	2000	2.47×10^{-7}	1.05×10^{-10}	175.45665	175.46074	2
2	-0.22	0.22	1.3	4.2	4.35×10^{-3}	6.24×10^{-11}	175.45665	175.46699	3
3	-0.22	0.22	1.35	4.15	1.54×10^{-2}	1.21×10^{-10}	175.46699	175.56718	6
4	-0.22	0.22	1.35	4.125	8.45×10^{-3}	9.03×10^{-11}	175.56718	175.62319	6
5	-0.22	0.22	1.35	4.1	1.81×10^{-2}	2.67×10^{-11}	175.56718	175.73421	8

TABLE 2

CONVERGENCE DATA FOR TRAJECTORY 5

Iteration	t_N	$\|\Psi\|$	$\|s\|$
0	175.56718	1.80×10^{-2}	1.96×10^{-5}
1	175.69711	1.07×10^{-2}	6.34×10^{-6}
2	175.66993	3.74×10^{-3}	1.12×10^{-6}
3	175.73047	4.20×10^{-4}	1.20×10^{-7}
4	175.73417	8.02×10^{-6}	5.05×10^{-10}
5	175.73421	3.71×10^{-7}	3.76×10^{-10}
6	175.73421	7.17×10^{-9}	3.74×10^{-12}
7	175.73421	1.30×10^{-10}	2.06×10^{-13}
8	175.73421	2.67×10^{-11}	5.07×10^{-14}

Figure 1 contains graphs of U_3 and U_4 for the unconstrained solution, Trajectory 1, and a constrained solution, Trajectory 4. An interesting feature of Trajectory 4 is the approximate "bang-bang" control for U_4. Figure 2 is a graph of the control variable U_4 for Trajectory 5, for Iterations 0 and 8.

Further numerical results appear in Reference 36.

Appendix B contains the description of a linear problem with a second order state-variable inequality constraint. The computational solution of this problem by Method 2 is given.

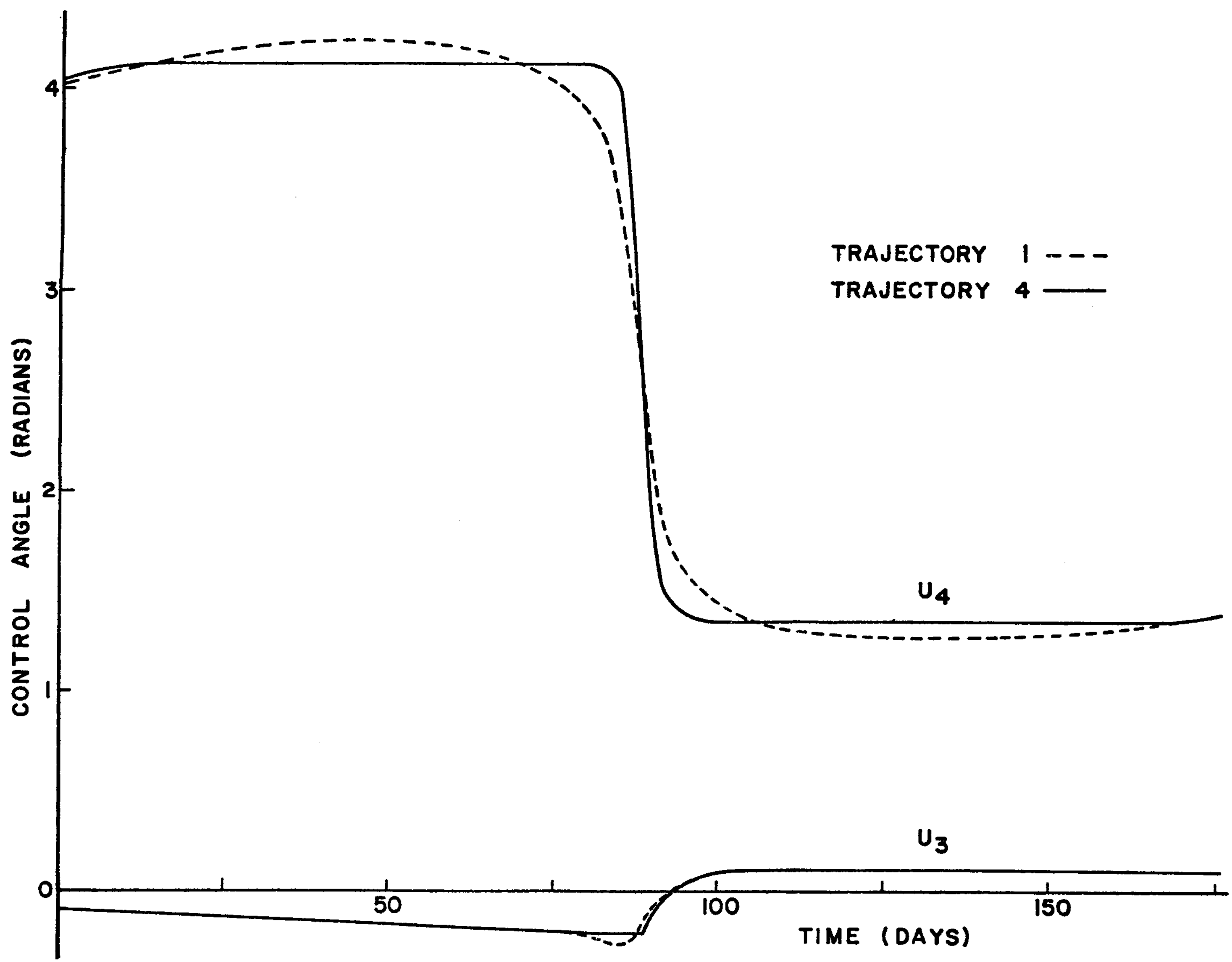


FIGURE 1 : U_3 AND U_4 FOR TRAJECTORIES 1 AND 4

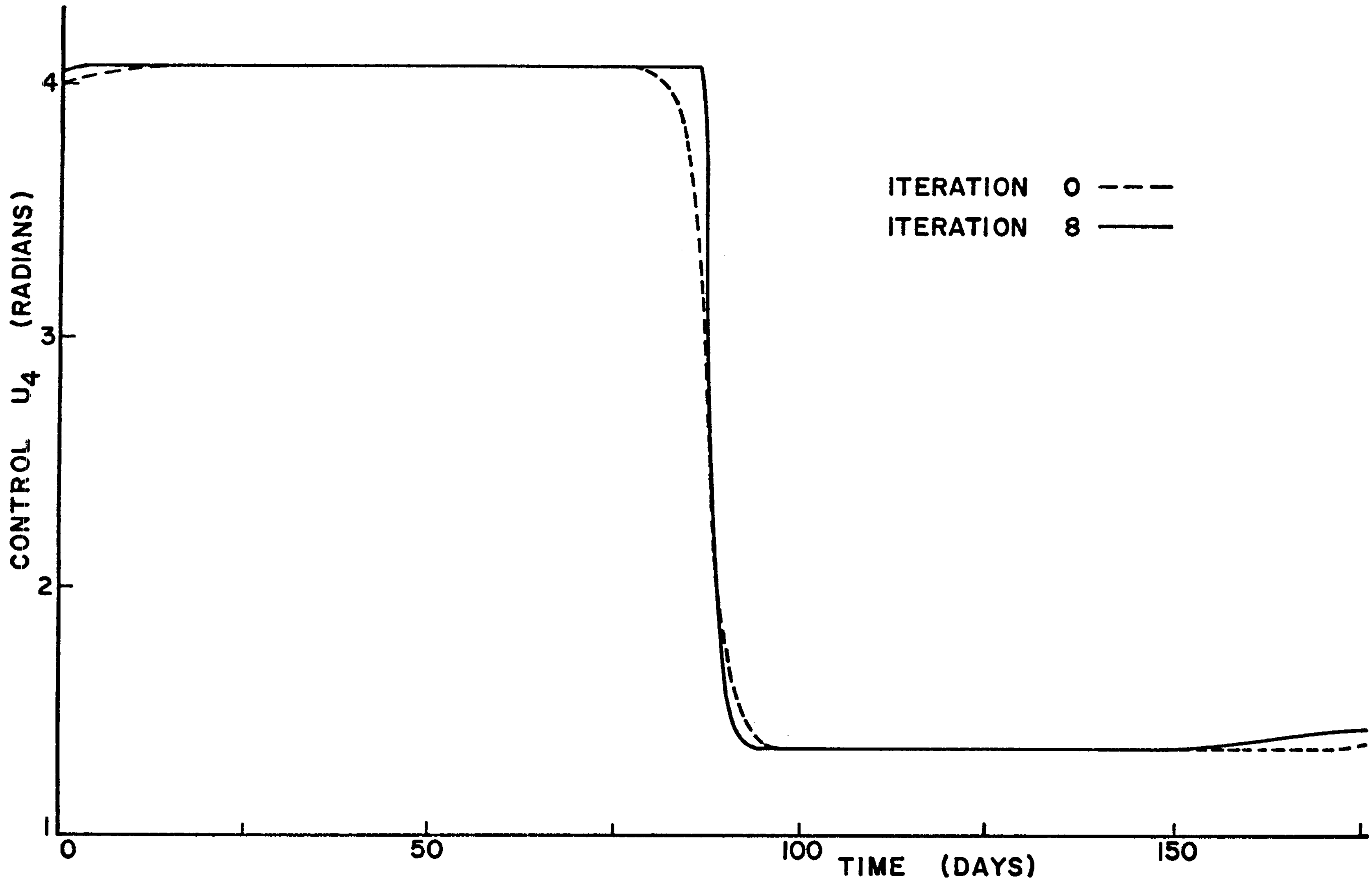


FIGURE 2 : CONTROL U_4 FOR TWO ITERATIONS (TRAJECTORY 5)

CHAPTER 6

CONCLUSIONS

The problem of optimization of nonlinear systems subject to inequality constraints has been investigated from the viewpoint of producing an optimal trajectory. Two general forms of inequality constraints were examined: $C(X,U,t)$, which explicitly involved the control, and $S(X,t)$, the state-variable constraint, which did not. It was shown that the control could be readily determined so that the trajectory would not cross a constraint boundary of the form $C(X,U,t) = 0$. On a state-variable constraint boundary, $S(X,t) = 0$, the control was chosen to satisfy $d^qS/dt^q = 0$, where the q-th derivative of S is the first one that explicitly contains the control. If the control is chosen in this manner, the derivatives d^jS/dt^j, $j < q$, must be zero at the point where the trajectory enters the boundary. Furthermore, the derivative $d^{q-1}S/dt^{q-1}$ must be zero at the point where the trajectory leaves the boundary. Thus, a state-variable constraint can be reduced to a constraint of the form $C(X,U,t)$, in addition to some intermediate boundary conditions.

A general problem involving inequality constraints, $C(X,U,t) \leq 0$, and intermediate boundary conditions was studied to obtain the relationships which govern its solution. It was found that whenever ρ of the constraints were simultaneously zero, then $\rho \leq m$, where m is the number of control variables, U_j. Furthermore, the $\rho \times m$ matrix, $[\partial C_{i_k}/\partial U_j]$, must be of full rank for $C_{i_k} = 0$, $k = 1,2,\ldots, \rho$. It was found that the Lagrange multiplier, P , which was used in the analysis, could be discontinuous at the point where the trajectory entered a state-variable constraint boundary, but could be continuous at the point where the trajectory left the boundary. This is an extension of the results of Bryson et al. [10] to the case of more than one control and more than one state-variable inequality constraint. Differential equations for the state X and the multiplier P were obtained in terms of the partial derivatives of a variational Hamiltonian, $H = Q + P^T F + M^T C$. The optimum control U and the multiplier M could be determined from a set of algebraic equations, in terms of X, P and t.

The various conditions which the solution to an inequality-constrained optimization problem must satisfy were restated in the form of a two-point boundary value problem. A new perturbation method for inequality-constrained problems was devised to handle the two-point boundary value problem. This method was based on the linearization of the differential equations for X and P, the optimality conditions giving M and U, and the terminal conditions, about a nominal trajectory, then calculating changes in the initial conditions and the terminal time so that the new nominal would more closely satisfy the terminal conditions. The resulting computational algorithm provided a rapidly converging procedure (if the initial approximation was "sufficiently close") for systems which are required to satisfy inequality constraints. This was demonstrated in the numerical experiments.

A simple test for the existence of points conjugate to the initial time t_0 was derived in Chapter 4. No conjugate point existed on the trajectory if a certain matrix was positive definite.

In an optimization or open-loop control problem the state variables and the control variables are obtained as functions of time: $X = X(t)$ and $U = U(t)$. The related problem, feedback or closed-loop control, gives the control as a function of the state: $U = U(X)$. The major difficulty in closed-loop control is to determine the entering- and exiting- corner times. McIntyre [41], discussing the closed-loop control problem associated with inequality-constrained systems, notes that near the corner points one would have to resort to open-loop control. A feedback control scheme based on the perturbation method of Chapter 4, where changes in the corner times are neglected, would probably give sufficiently accurate results. Further work on this topic is required.

APPENDICES

APPENDIX A

A study of Earth-Mars transfer trajectories is important because of the expected expeditions (manned or unmanned) to Mars within the two decades following 1970. Some of the space vehicles on these missions may be powered by low-thrust ion or plasma jet engines. Such engines are characterized by low fuel consumption and continuous thrusting capability. At a low-thrust level the acceleration of the vehicle will be small and therefore the thrust may be applied for most or all of the mission. For the problem studied by Fowler [20], the thrust magnitude and the mass-flow rate were taken as constants. The mathematical model is given below.

A low-thrust Earth-Mars trajectory is sought. The vehicle is assumed to travel in an inverse square gravitational field. The orbit of Mars is assumed to be an ellipse with an eccentricity of $e = 0.093393$ and a semi-major axis of $a = 1.523691$ AU (astronomical units). The orbit of Mars is assumed to lie in an plane which is inclined to the ecliptic at an angle of $i = 0.032289$ radians; see Figure A1. The equations of motion which describe the transfer trajectory are expressed in a heliocentric rectangular cartesian coordinate system whose X-axis coincides with the line of ascending node for the Mars orbit. The Y-axis lies in the Ecliptic plane and the Z-axis coincides with the angular momentum vector of the earth with respect to the sun. The coordinate system is shown in Figure A1.

Considering the sun as a homogeneous, spherical body with a gravitational potential given by

$$\mathbb{P} = \gamma m/r \qquad \text{(A.1)}$$

where r is the distance from the sun to the position of the vehicle, m is the mass of the vehicle, and γ is the solar gravitational constant (γ = 0.000296007536 AU^3/day^2), then the motion of the vehicle in the gravitational field of the sun is given by

$$m\dot{v} = \nabla\mathbb{P} + T \qquad \text{(A.2)}$$

where v is the vehicle velocity vector, and T is the vehicle thrust vector. The solar radiation forces and drag forces have been neglected. The thrust, T, is given by

$$T = -\beta c \qquad \text{(A.3)}$$

where β is the propellant mass-flow rate, and c is the effective propellant exhaust velocity relative to the vehicle. The components of the thrust vector in the (X,Y,Z)-coordinate system are specified by two thrust orientation angles, Ψ and Θ, as shown in Figure A2. Letting (U,V,W) and (X,Y,Z) be the velocity and position components respectively in the (X,Y,Z)-coordinate system, the equations of motion become

$$\dot{U} = -\frac{\gamma X}{r^3} + \frac{\beta|c|}{m}[\cos\Theta\cos\Psi] \tag{A.4}$$

$$\dot{V} = -\frac{\gamma Y}{r^3} + \frac{\beta|c|}{m}[\cos\Theta\sin\Psi] \tag{A.5}$$

$$\dot{W} = -\frac{\gamma Z}{r^3} + \frac{\beta|c|}{m}[\sin\Theta] \tag{A.6}$$

$$\dot{X} = U \tag{A.7}$$

$$\dot{Y} = V \tag{A.8}$$

$$\dot{Z} = W \tag{A.9}$$

where $r^2 = X^2 + Y^2 + Z^2$, and $|c|$ is the magnitude of c. The mass m satisfies the differential equation

$$\dot{m} = -\beta \ . \tag{A.10}$$

Since β is a constant,

$$m = m_0 - (t-t_0)\beta \ . \tag{A.11}$$

For the units chosen in the problem,

time is in days

position is in AU

speed is in AU/day

mass is in vehicle mass; and

$m_0 = 1$

$t_0 = 0$ (12:00 noon, May 9, 1971)

$\beta = 0.00108$ vehicle mass/day

$|c| = 0.0453649854$ AU/day

At t_0: the argument of perihelion of Mars is $\omega = 5.8541335$ radians, and its eccentric anomaly is 4.250885 radians.

The position and velocity of Mars are computed by finding the eccentric anomaly, E, as a function of t. The eccentric anomaly is given by Kepler's equation,

$$E - e\sin E = \left[\frac{\gamma}{a^3}\right]^{1/2}(t-t_0) + E_0 - e\sin E_0 \qquad (A.12)$$

where
- e is the eccentricity of Mars' orbit,
- E_0 is the eccentric anomaly at t_0,
- γ is the solar gravitational constant, and
- a is the semi-major axis of Mars' orbit.

With E known, X'', Y'' and Z'' can be calculated from

$$X'' = a(\cos E)$$

$$Y'' = a\sqrt{1-e^2}(\sin E)$$

$$Z'' = 0$$

where the (X'',Y'',Z'')-coordinate system is shown in Figure A1. The coordinate transformation from the (X'',Y'',Z'')-system to the (X,Y,Z)-system is given by the equation,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \cos i(\sin\omega) & \cos\omega(\cos i) & -\sin i \\ \sin i(\sin\omega) & \sin i(\cos\omega) & \cos i \end{bmatrix} \begin{bmatrix} X''-ae \\ Y'' \\ Z'' \end{bmatrix} \quad (A.13)$$

where ω is the argument of perihelion of Mars, i is the angle of inclination of Mars' orbital plane. The initial conditions are

$$U(t_0) = -0.0003455906 \text{ AU/day} \quad (A.14)$$

$$V(t_0) = -0.0171986836 \text{ AU/day} \quad (A.15)$$

$$W(t_0) = 0.0 \text{ AU/day} \quad (A.16)$$

$$X(t_0) = -0.9998 \text{ AU} \quad (A.17)$$

$$Y(t_0) = 0.02009 \text{ AU} \tag{A.18}$$

$$Z(t_0) = 0.0 \text{ AU} . \tag{A.19}$$

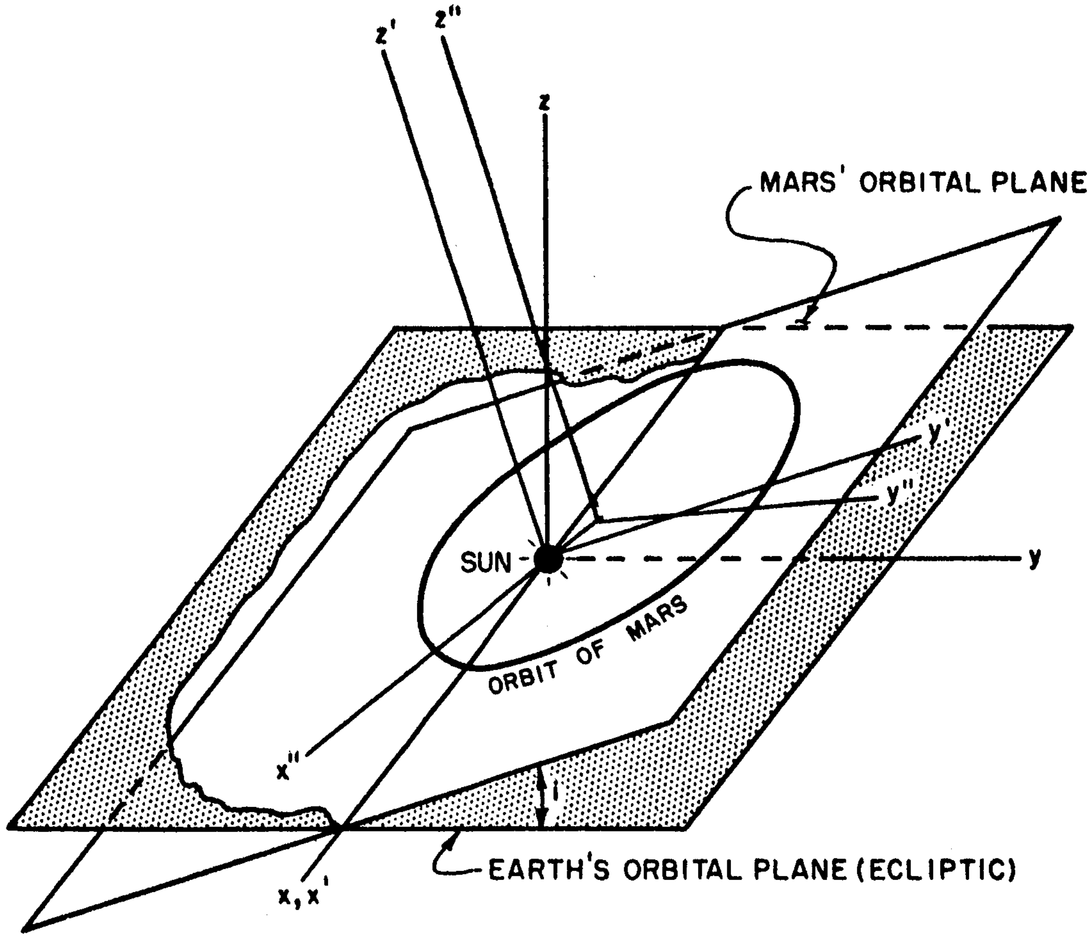


FIGURE A1. COORDINATE SYSTEMS

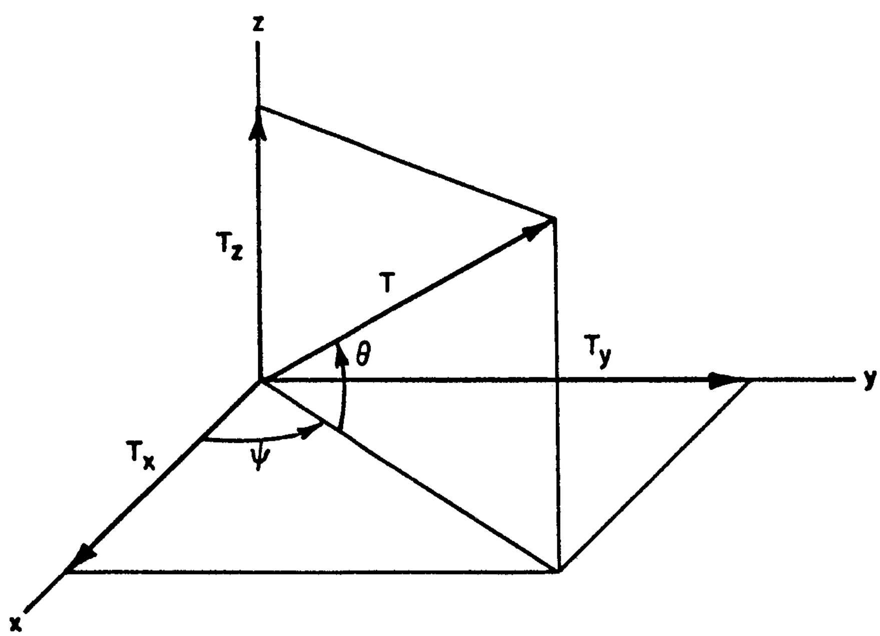


FIGURE A2. THRUST VECTOR COMPONENTS

APPENDIX B

As an example, the computational procedure of Chapter 4 is given for a state-variable inequality constraint. This problem was first discussed by Bryson *et al*. [10].

The problem concerns the minimization of

$$\int_0^1 \tfrac{1}{2}U^2 dt$$

subject to

$$\dot{X}_1 = U$$

$$\dot{X}_2 = X_1$$

$$S = X_2 - 0.1 \leq 0$$

and

$$X_1(0) = 1$$

$$X_2(0) = 0$$

$$X_1(1) = -1$$

$$X_2(1) = 0 \ .$$

The first derivative of S which explicitly contains the control U is $\ddot{S} = U$. Thus, at the point $t = t_1$, where the trajectory meets the constraint boundary,

$$L_1^{(1)} = \dot{S} = \dot{X}_2 = X_1 = 0$$

$$L_2^{(1)} = S = X_2 - 0.1 = 0 \ .$$

The variational Hamiltonian is

$$H = \begin{cases} \frac{1}{2}U^2 + P_1U + P_2X_1 \ ; & S < 0 \\ \\ \frac{1}{2}U^2 + P_1U + P_2X_1 + M\ddot{S} \ ; & S = 0 \ . \end{cases}$$

The differential equations for the multipliers P_1 and P_2 are

$$\dot{P}_1 = - P_2$$
$$\dot{P}_2 = 0 \ ;$$

The boundary conditions at t_1 are given by Equations (3.24) and (3.25);

$$P_1(t_1^+) = P_1(t_1^-) - \nu_1^{(1)}$$

$$P_2(t_1^+) = P_2(t_1^-) - \nu_2^{(2)}$$

$$H(t_1^-) = \frac{1}{2}P_1^2(t_1^-) = 0 \ .$$

Thus at $t = t_1^-$, $P_1(t_1^-) = 0$. Note that both P_1 and P_2 are discontinuous at t_1. The problem can be split into two segments; one from $0 \le t \le t_1^-$ and the other from $t_1^+ \le t \le 1$.

Part 1. $0 \le t \le t_1^-$.

The unknowns to be determined are $P_1(0)$, $P_2(0)$ and t_1. The boundary conditions are

$$L_1^{(1)} = X_1(t_1^-) = 0$$

$$L_2^{(1)} = X_2(t_1^-) - 0.1 = 0$$

$$L_3^{(1)} = P_1(t_1^-) = 0 \ .$$

The linearized differential equations are

$$\begin{aligned} \dot{x}_1 &= -p_1 \\ \dot{x}_2 &= x_1 \\ \dot{p}_1 &= -p_2 \\ \dot{p}_2 &= 0 \end{aligned}$$

so that

$$\Phi_1(t) = \begin{bmatrix} -t & \frac{t^2}{2} \\ -\frac{t^2}{2} & \frac{t^3}{6} \end{bmatrix}$$

$$\Phi_2(t) = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}.$$

The linearized terminal conditions are

$$\begin{bmatrix} x_1(t_1^-) \\ x_2(t_1^-) \\ p_1(t_1^-) \\ p_2(t_1^-) \end{bmatrix} = \begin{bmatrix} 0 & -\dot{X}_1(t_1^-) & 1 & 0 & 0 \\ 0 & -\dot{X}_2(t_1^-) & 0 & 1 & 0 \\ 0 & -\dot{P}_1(t_1^-) & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ \Delta t_1 \\ \Delta L_1^{(1)} \\ \Delta L_2^{(1)} \\ \Delta L_3^{(1)} \end{bmatrix}$$

so that the corrections $p_1(0)$, $p_2(0)$ and Δt_1 are given by the solution to

$$\begin{bmatrix} 0 & -\dot{X}_1(t_1^-) & t_1 & -\frac{t_1^2}{2} \\ 0 & -\dot{X}_2(t_1^-) & \frac{t_1^2}{2} & -\frac{t_1^3}{6} \\ 0 & -\dot{P}_1(t_1^-) & -1 & t_1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_2 \\ \Delta t_1 \\ p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} -\Delta L_1^{(1)} \\ -\Delta L_2^{(1)} \\ -\Delta L_3^{(1)} \\ 0 \end{bmatrix}$$

The computational procedure is the following:

(a) Guess $P_1(0)$, $P_2(0)$ and t_1

(b) Integrate the differential equations for X_1 , X_2 , P_1 , P_2 from 0 to t_1

(c) Compute $\Phi_1(t_1^-)$ and $\Phi_2(t_1^-)$

(d) Calculate $p_1(0)$, $p_2(0)$ and Δt_1

(e) Form new values of $P_1(0)$, $P_2(0)$ and t_1 .

Part 2. $t_1^+ \le t \le 1$.

The unknowns to be determined are $P_1(t_1^+)$ and $P_2(t_1^+)$. The boundary conditions are

$$X_1(t_1^+) = 0 \qquad X_2(t_1^+) = 0.1$$

$$L_1^{(2)} = X_1(1) + 1 = 0$$

$$L_2^{(2)} = X_2(1) = 0 \ .$$

On the boundary $S = 0$ $(t_1^+ \le t \le t_2)$ the linearized differential equations are

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = x_1$$
$$\dot{p}_1 = -p_2$$
$$\dot{p}_2 = 0$$

so that

$$\Phi_1(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Phi_2(t) = \begin{bmatrix} 1 & t_1-t \\ 0 & 1 \end{bmatrix}.$$

For $t \geq t_2$ the linearized differential equations are

$$\dot{x}_1 = -p_1$$

$$\dot{x}_2 = x_1$$

$$\dot{p}_1 = -p_2$$

$$\dot{p}_2 = 0$$

and therefore

$$\Phi_1(t) = \begin{bmatrix} t_2-t & v(t) \\ -\frac{1}{2}(t_2-t)^2 & w(t) \end{bmatrix}$$

$$v(t) = -\frac{1}{2}(t_1-t)^2 + \frac{1}{2}(t_1-t_2)^2$$

$$w(t) = \frac{1}{6}(t_1-t)^3 + \frac{1}{2}(t_1-t_2)^2 t - \frac{1}{6}(t_1-t_2)^2(2t_2+t_1)$$

$$\Phi_2(t) = \begin{bmatrix} 1 & t_1-t \\ 0 & 1 \end{bmatrix}.$$

The linearized boundary conditions are

$$x_1(1) = \Delta L_1^{(2)}$$

$$x_2(1) = \Delta L_2^{(2)}$$

so that

$$\begin{bmatrix} t_2-1 & v(1) \\ -\frac{1}{2}(t_2-1)^2 & w(1) \end{bmatrix} \begin{bmatrix} p_1(t_1^+) \\ p_2(t_1^+) \end{bmatrix} = \begin{bmatrix} \Delta L_1^{(2)} \\ \Delta L_2^{(2)} \end{bmatrix}.$$

The computational procedure is similar to that of Part 1: only need to estimate $P_1(t_1^+)$ and $P_2(t_1^+)$ since t_1 is known.

The optimum values are

$$P_1(0) = 6.666666666$$

$$P_2(0) = 22.22222222$$

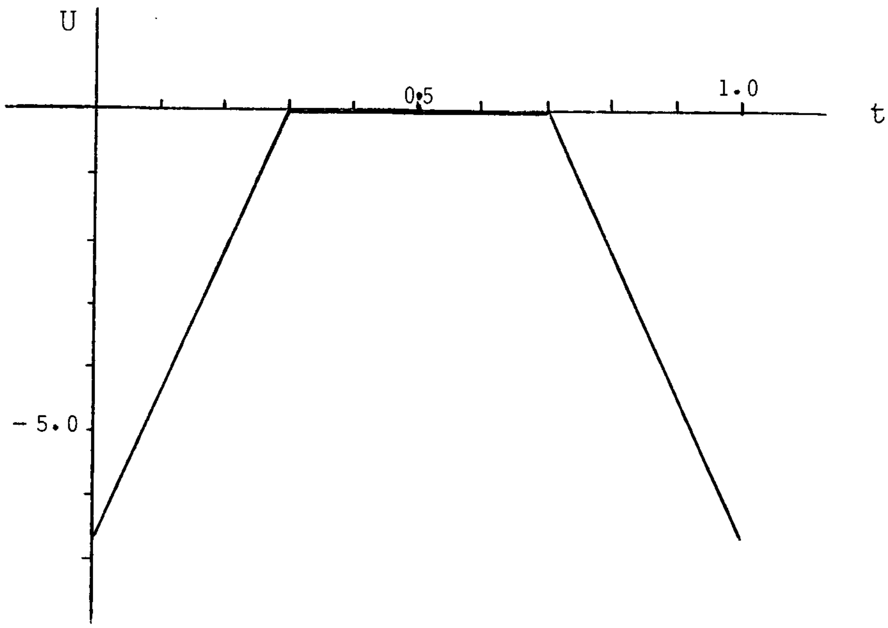
$$P_1(t_1^+) = -8.888888888$$

$$P_2(t_1^+) = -P_2(0)$$

$$t_1 = 0.3$$

$$t_2 = 0.7 \quad .$$

The optimum control is



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VITA

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